Analysis I
Terrence Tao
newell.jensen@gmail.com

Chapter 2 - Starting at the beginning: the natural numbers

Exercises:

§2.2 Addition

2.2.1. Prove Proposition 2.2.5. (Hint: fix two of the variables and induct on the third.)

Proof. Proposition 2.2.5 claims

For any natural numbers a,b,c, we have $(a+b)+c = a+(b+c)$.

We use induction. Let a, b, c, be natural numbers.

base case: For $c = 0$, $(a+b)+c = a+(b+c)$ becomes

$$
\begin{aligned}
(a+b)+0 &= a+(b+0) && [c=0] \\
(a+b)+0 &= a+(b) && [\text{Lemma 2.2.2.}] \\
(a+b) &= a+(b) && [\text{Definition 2.2.1 and Lemma 2.2.2.}] \\
a+b &= a+b
\end{aligned}
$$

induction hypothesis: Suppose that $(a+b)+c = a+(b+c)$.

induction step: We must now prove that $(a+b)+(c{++}) = a+(b+(c{++}))$.

The left-hand side $(a+b)+(c{++})$ becomes

$$(a+b)+(c{++}) = ((a+b)+c){++} \qquad [\text{Lemma 2.2.3.}]$$

while the right-hand side $a+(b+(c{++}))$ becomes

$$
\begin{aligned}
a+(b+(c{++})) &= a+((b+c){++}) && [\text{Lemma 2.2.3.}] \\
&= (a+(b+c)){++} && [\text{Lemma 2.2.3.}]
\end{aligned}
$$

Thus, $((a+b)+c){++} = (a+(b+c)){++}$ by the induction hypothesis and we have closed the induction. □

2.2.2. Prove Lemma 2.2.10. (Hint: use induction.)

Proof. Lemma 2.2.10 claims,

Let a be a positive number. Then there exists exactly one natural number b such that $b{++} = a$.

We use induction on a.

base case: Let $a = 1$, with is a positive number. We know that $0{++} = 1$. 0 is a natural number by Axiom 2.1 and is unique by Proposition 2.1.6.

induction hypothesis: Suppose that $a = n$ and that there exists exactly one natural number b such that $b{++} = a$.

induction step: We must now prove that for $a = n{++}$ that there exists exactly one natural number b such that $b{++} = a$.

For $a = n{++}$, we know that $n{++} = (b{++}){++}$ by the induction hypothesis. From Axiom 2.2 and the induction hypothesis, we know that $b{++}$ is a natural number. From Proposition 2.1.8 we know that no natural number is equal to its successor and therefore we must have that $b \neq b{++}$, showing that $b{++}$ is unique such that $(b{++}){++} = a$. This closes the induction. ☐

2.2.3. Prove Proposition 2.2.12. (Hint: you will need many of the preceding propositions, corollaries, and lemmas.)

Proof. Proposition 2.2.12 claims,

Let a,b,c be natural numbers. Then

(a) (*Order is reflexive*) $a \geq a$.

(b) (*Order is transitive*) *If* $a \geq b$ *and* $b \geq c$, *then* $a \geq c$.

(c) (*Order is anti-symmetric*) *If* $a \geq b$ *and* $b \geq a$, *then* $a = b$.

(d) (*Addition preserves order*) $a \geq b$ *if and only if* $a + c \geq b + c$.

(e) $a < b$ *if and only if* $a{++} \leq b$.

(f) $a < b$ *if and only if* $b = a + d$ *for some* positive *number* d.

(a). Since $a = a$, by definition, then by Definition 2.2.11 we must have that $a \geq a$.

(b). If $a \geq b$ and $b \geq c$, then by Definition 2.2.11 we must have that $a = b + n_1$ and $b = c + n_2$ for some natural numbers n_1 and n_2, respectively. Then $a = (c+n_2)+n_1 = c+(n_2+n_1)$, since addition is associative (Proposition 2.2.5). Furthermore, $(n_2 + n_1)$ is a natural number from the definition of addition (Definition 2.2.1) and therefore, from Definition 2.2.11, we have that $a \geq c$.

(c). If $a \geq b$ and $b \geq a$, then by Definition 2.2.11 we must have that $a = b + n_1$ and $b = a + n_2$ for some natural numbers n_1 and n_2, respectively. Then $a = (a+n_2)+n_1 = a+(n_2+n_1)$ since addition is associative (Proposition 2.2.5). From Lemma 2.2.2, we must have that $(n_2 + n_1) = 0$, and therefore, from the previous equalities, we have that $a = b$.

(d). Note: For *iff* type proofs we must prove both directions.

($\rightarrow$): If $a \geq b$, then by Definition 2.2.11 we must have that $a = b + n$ for some natural number n. Let c be some natural number. Then $a + c = b + n + c = b + c + n$ (addition is commutative), so that by Definition 2.2.11 we have that $a + c \geq b + c$.

($\leftarrow$): If $a+c \geq b+c$ then by Definition 2.2.11 we must have that $a+c = b+c+n$ for some natural number n. This can be re-written as $a+c = (b+n)+c$ since addition is commutative and associative. Then using the cancellation law, Proposition 2.2.6, we see that $a = b+n$ and therefore, by Definition 2.2.11, we have that $a \geq b$.

(e.)

($\rightarrow$): If $a < b$ then by Definition 2.2.11 we must have that $b \geq a$ and $a \neq b$. Therefore, we know that $b = a+n$ for some natural number n and since $a \neq b$ we see that n must be a positive number (if it was 0 we would have a contradiction). From Lemma 2.2.10 there exists exactly one natural number, say c, such that $c{+}{+} = n$. Thus, we have $b = a + (c{+}{+})$ and from Lemma 2.2.3 we see that this can be formulated as $b = (a+c){+}{+}$. Using the commutativity of addition this becomes $b = (c+a){+}{+}$ and then using Lemma 2.2.3 again, we conclude that $b = c + (a{+}{+})$ and using commutativity one last time we have that $b = (a{+}{+}) + c$. Therefore, by Definition 2.2.11, we have that $a{+}{+} \leq b$.

($\leftarrow$): If $a{+}{+} \leq b$, then by Definition 2.2.11, we have that $b = (a{+}{+}) + n$ for some natural number n. From Definition 2.2.1, we can formulate this as $b = (a+n){+}{+}$ and then since addition is commutative this becomes $b = (n+a){+}{+}$. Then we see from Lemma 2.2.3, we have $b = a + (n{+}{+})$ and from Axiom 2.2 we know that $n{+}{+}$ is a natural number. Thus, from Defintiion 2.2.11 we see that $b = a + (n{+}{+})$ implies $a \leq b$.

(f.)

($\rightarrow$): If $a < b$ then by *(e)* we must have that $a{+}{+} \leq b$. By Definition 2.2.11, this means that $b = (a{+}{+}) + n$ for some natural number n. From Definition 2.2.1 this becomes $b = (a+n){+}{+}$ and by Lemma 2.2.3 we then have $b = a + (n{+}{+})$. From Axiom 2.2 we know that $(n{+}{+})$ is some positive natural number, say d, from Axiom 2.3. Therefore, $b = a + d$ for some positive number d.

($\leftarrow$): If $b = a + d$ for some positive number d, then d must be the successor of some natural number, say n. Thus we see that $b = a + (n{+}{+})$ and by Lemma 2.2.3 we have $b = (a+n){+}{+}$. From commutativity of addition and Lemma 2.2.3 this then becomes $b = (a{+}{+}) + n$ which by Definition 2.2.11 implies that $a{+}{+} \leq b$. Since $b = a + d$ for some positive number d, we must have that $b \neq a$ by Lemma 2.2.2. Therefore, by Definition 2.2.11 we can conclude that $a < b$. □

2.2.4. Justify the three statements marked (why?) in the proof of Proposition 2.2.13. Let us list the (why?)s:

1. *When* $a = 0$ *we have* $0 \leq b$ *for all* b.

   Any natural number b must either be equal to 0 or equal to some multiple successor of 0, which by the ordering of the natural numbers (Definition 2.2.11) is larger than 0.

2. *If* $a > b$, *then* $a{+}{+} > b$.

   Even though it hasn't been introduced in the textbook yet, instead of proving with the Propositions and Lemmas we will justify this claim with a proof by contrapositive as it the simplicity seems to be calling for it.

   If $a{+}{+} \not> b$ then we must have that $a{+}{+} \leq b$ from the ordering of the natural numbers (Definition 2.2.11). From *(e)* of Exercise 2.2.3, we then see that we must have $a < b$ which shows that $a \not> b$.

   If this seems too hand *wavy* at this point in our cultivation of the foundations of mathematics for your liking, let us justify this claim without using a proof by contrapositive.

   First we will prove a corollary from Lemmas 2.2.2 and 2.2.3:

From Lemma 2.2.3, if we set $m = 0$ we have $n + (0{++}) = (n + 0){++}$ and by Lemma 2.2.2 and the fact that $0{++} = 1$, we then have $n + 1 = n{++}$. This shows, using Definition 2.2.11, that $n{++} > n$.

Now to the proof. If $a > b$, then from Definition 2.2.11 we know that $a \neq b$ and $a = b + n$ for some positive natural number n. Using the corrollary above we have $a{++} > a > b$ and therefore $a{++} > b$. For the ones that would like even more proof of this, the last step can be seen by noting that $a{++} = a + 1$ and using the fact that $a = b + n$ so we must have $a{++} = (b + n) + 1 = b + (n + 1)$. From Definition 2.2.11, the result follows.

3. *If* $a = b$, then $a{++} > b$.

   If $a = b$ then from the collorary derived in (2) above, we know that for a natural number n that $n{++} > n$. Therefore, we must have that $b{++} > b$. However, since $a = b$ we see that $a{++} = b{++}$ and therefore, we conclude that $a{++} > b$.

2.2.5. Prove Proposition 2.2.14. (Hint: define $Q(n)$ to be the property that $P(m)$ is true for all $m_0 \leq m < n$; note that $Q(n)$ is vacuously true when $n < m_0$.)

Proof. We are going to prove the Strong principle of induction using standard induction. Proposition 2.2.14 claims:

Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.

As the hint suggests, let us define $Q(n)$ to be the property that $P(m)$ is true for all $m_0 \leq m < n$. We prove this by induction on n.

base case: Let $n = 0$ so that we have $Q(0)$ which is the property that '*if $P(m)$ is true for all $m_0 \leq m < 0$, then $P(0)$ is true*'. This is the vacuously true, as there are no natural numbers less than 0, therefore $P(0)$ is true and subsequently so is $Q(0)$.

induction hypothesis: Suppose $Q(n)$ is true, that is, the property that '*if $P(m)$ is true for all $m_0 \leq m < n$, then $P(n)$ is also true*'.

induction step: $Q(n + 1)$ is the claim that '*if $P(m)$ is true for all $m_0 \leq m < n + 1$, then $P(n + 1)$ is also true*'. If $m < n + 1$, from Definition 2.2.11, we conclude that $n + 1 \neq m$ and there is a positive natural number, say d, such that $n + 1 = m + d$. Since d is an arbitrary positive natural number, let $d = 1$. The reason we can do this is because the criteria of $n + 1 \neq m$ and d being a positive natural number are both satsified by setting $d = 1$, i.e., no rules have been broken. We then arrive at $n + 1 = m + 1$ and using the cancellation law of Proposition 2.2.6, we then have $n = m$. Thus, the inequality $m_0 \leq m < n + 1$ becomes $m_0 \leq n < n + 1$ and from the induction hypothesis we know that $P(n)$ is true and therefore we must have that $P(n + 1)$ is true. Thus, $Q(n + 1)$ is true. This closes the induction. □

2.2.6. Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{++})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$; this is known as the *principle of backwards induction*. (Hint: apply induction to the variable n).

Proof. We are given a property $P(m)$ such that whenever $P(m{++})$ is true, then $P(m)$ is true. We are also given that n is a natural number and that $P(n)$ is also true. We are to prove that $P(m)$ is true for all natural numbers $m \leq n$. We use induction on n.

base case: Let $n = 0$ and suppose that $P(0)$ is true. Then we trivially have that $P(m)$ is true for all natural numbers $m \le 0$.

induction hypothesis: Suppose that $P(n)$ is true such that $P(m)$ is true for all natural numbers $m \le n$.

induction step: Suppose that $P(n+1)$ is true. We now need to prove that $P(m)$ is true for all natural numbers $m \le n+1$. $P(m)$ once again is the property that if $P(m{+}{+})$ is true, then $P(m)$ is true. Since $P(n+1)$ is true by hypothesis, then $P(n)$ is true. Since $P(n)$ is true, we must have that $P(m)$ is true for all natural numbers $m \le n$ by the induction hypothesis. Therefore, we have that $P(m)$ is true for $m \le n$ as well as for $P(n+1)$, i.e., for $n+1$. Thus, $P(m)$ is true for all natural numbers $m \le n+1$. This closes the induction. □

§2.3 Multipication

2.3.1 Prove Lemma 2.3.2. (Hint: modify the proofs of Lemmas 2.2.2, 2.2.3 and Proposition 2.2.4.)

Proof. Lemma 2.3.2 claims,

Let n, m be natural numbers. Then $n \times m = m \times n$.

First, we modify the proofs of Lemmas 2.2.2 and 2.2.3. We will then use this to modify the proof of Proposition 2.2.4.

[*Modification of Lemma* 2.2.2 — proving that $0 \times m = 0$] We use induction on n:

base case: The base case $0 \times 0 = 0$ follows since we know that $0 \times m = m$ for every natrual number m, and 0 is a natural number.

induction hypothesis: Suppose that $n \times 0 = n$.

induction step: We must show that $(n{+}{+}) \times 0 = n{+}{+}$. By definition of multiplication we know that $(n{+}{+}) \times 0 = (n \times 0){+}{+}$, which is equal to $n{+}{+}$ since $n \times 0 = n$ from the induction hypothesis. This closes the induction.

[*Modification of Lemma* 2.2.3 — proving that $n \times (m{+}{+}) = n + (n \times m)$] We use induction on n:

base case: For the base case $n = 0$ we have to prove that $0 \times (m{+}{+}) = 0 + (0 \times m)$. We know that for any natural number m, that $m{+}{+}$ is also a natural number and by the definition of multiplication we know that $0 \times (m{+}{+}) = 0$. Therefore the left-hand side of the equation is equal to 0. By the definition of multiplication the right-hand side $0 + (0 \times m)$ becomes $0 + (0)$ which equals 0. Thus the left and right-hand sides are equal.

induction hypothesis: Suppose that for any natural numbers n, m that $n \times (m{+}{+}) = n + (n \times m)$.

induction step: We must prove that $(n+1) \times (m{+}{+}) = (n+1) + ((n+1) \times m)$. Noting that $n + 1 = n{+}{+}$ this can be re-written as $(n{+}{+}) \times (m{+}{+}) = (n{+}{+}) + ((n{+}{+}) \times m)$.

The left-hand side is $(n \times m{+}{+}) + (m{+}{+})$ by definition of multiplication, which is equal to $n + (n \times m) + (m{+}{+})$ by the induction hypothesis. By the definition of addition we see that this can be further simplified to $(n \times m) + (m + n){+}{+}$.

Similarly for the right-hand side, by the definition of multiplication we have $(n{+}{+}) + (n \times m) + m$. This can be further simplified to $(n \times m) + (n + m){+}{+}$ and finally $(n \times m) + (m + n){+}{+}$ from the commutativity of addition. As both sides of the equation are equal, this closes the induction.

[*Modification of Proposition* 2.2.4 — proving that $n \times m = m \times n$] We use induction on n (keeping m fixed):

base case: Let $n = 0$. From the definition of multiplication and the modification of Lemma 2.2.2 above, $0 \times m = 0$ and $m \times 0 = 0$. Therefore, $0 \times m = m \times 0$ for natural numbers n and m.

induction hypothesis: Suppose that $n \times m = m \times n$ for natural numbers n and m.

induction step: We must prove that $(n + 1) \times m = m \times (n + 1)$ for natural numbers n and m. Let us re-write this as $(n{+}{+}) \times m = m \times (n{+}{+})$. The left-hand side is $(n \times m) + m$ by the definition of multiplication while the righ-hand side is $m + (m \times n)$ from the modification of Lemma 2.2.3 above. Using the induction hypothesis, we see that both the left-hand and right-hand sides are equal, closing the induction.

Therefore, $n \times m = m \times n$ showing that multiplication is commutative. □

2.3.2 Prove Lemma 2.3.3. (Hint: prove the second statement first.)

Proof. Lemma 2.3.3 claims,

Let n, m be natural nubmers. Then $n \times m = 0$ if and only if at least one of n, m is equal to zero. In particular, if n and m are both positive, then nm is also positive.

($\leftarrow$): If at least one of n, m is equal to zero then $n \times m$ either takes the form $0 \times m$ or $n \times 0$, which are both equal to zero by the definition of multipication and modification of Lemma 2.2.2 as seen in the last exercise. Therefore, $n \times m = 0$.

($\rightarrow$): If $n \times m = 0$ then since $0 \times m = 0$ by the definition of multiplication we see that $n = 0$ satisfies this equality. However we also know that since $n \times 0 = 0$ by the modification of Lemma 2.2.2 above, we see that $m = 0$ also satisfies this equality. Therefore, at least one of n, m is equal to zero.

In particular, if $n > 0$ and $m > 0$ then $n \geq 1$ and $m \geq 1$. Suppose that $n = 1$. Then $n \times m = 1 \times m = 0 + m = m$ by the definition of multipliction and addition. Since multiplication is commutative this argument also holds for $m = 1$. Either way, we see that nm is also positive. □

2.3.3 Prove Proposition 2.3.5. (Hint: modify the proof of Proposition 2.2.5 and use the distributive law.)

Proof. Proposition 2.3.5 claims,

For any natural nubmers a, b, c, we have $(a \times b) \times c = a \times (b \times c)$.

We keep a and b fixed and use induction on c.

base case: For $c = 0$ we have $(a \times b) \times 0 = a \times (b \times 0)$ and both sides are zero.

induction hypothesis: Suppose that $(a \times b) \times c = a \times (b \times c)$.

induction step: We will prove that $(a \times b) \times (c{+}{+}) = a \times (b \times (c{+}{+}))$.

On the left-hand side $(a \times b) \times (c{+}{+})$ is $(c{+}{+}) \times (a \times b)$ from commutativity and then $(c \times (a \times b)) + (a \times b)$ from the definition of multiplication. One more operation of commutativity gives us $((a \times b) \times c) + (a \times b)$.

On the right-hand side $a \times (b \times (c{++}))$ is $a \times ((c{++}) \times b)$ from commutativity and then $a \times ((c \times b) + b)$ from the definition of multiplication. Using Proposition 2.3.4 this becomes $(a \times (c \times b)) + (a \times b)$ which is then $(a \times (b \times c)) + (a \times b)$ from commutativity.

From the induction hypothesis, we see that the left-hand and right-hand sides are both equal, closing the induction. □

2.3.4 Prove the identity $(a + b)^2 = a^2 + 2ab + b^2$ for all natural numbers a, b.

Proof. Let a, b be natural numbers.

Expanding the left-hand side we have $(a + b)(a + b)$ which becomes $(a + b)a + (a + b)b$ from the distributive law. Using the distributive law once again this becomes $aa + ba + ab + bb$ which is equal to $a^2 + 2ab + b^2$ since multiplication is commutative. □

2.3.5 Prove Proposition 2.3.9. (Hint: fix q and induct on n.)

Proof. Proposition 2.3.9 claims,

Let n be a natural number, and let q be a positive number.
Then there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.

We fix q and use induction on n.

base case: Let $n = 0$. Then $n = mq + r$ becomes $0 = mq + r$. Let $m = 0$ and $r = 0$ as zero is a natural number and $r = 0$ satisfies $0 \leq r < q$. The right-hand side becomes $0q + 0 = 0 + 0 = 0$ showing that there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.

induction hypothesis: Suppose that there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq+r$.

induction step: We will prove that there exist natural numbers m, r such that $0 \leq r < q$ and $n+1 = mq+r$.

Note that $n + 1 = mq + r$ becomes $(m_1 q + r_1) + 1 = mq + r$ from the induction hypothesis where we know that m_1, r_1 are natural numbers. This becomes $m_1 q + (r_1 + 1)$ from associativity of addition and we know that $r_1 + 1 = r_1{++}$, which is a natrual number. Therefore, we see that if we let $m = m_1$ and $r = r_1{++}$ the left-hand and right-hand sides are equal. This shows that there exist natural numbers m, r such that $0 \leq r < q$ and $n + 1 = mq + r$, closing the induction. □