Analysis I
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Chapter 3 - Set theory

Exercises:

§3.1 Fundamentals

3.1.1. Show that the definition of equality in Definition 3.1.4 is reflexive, symmetric, and transitive.

Proof.

Reflexive: Every element of A is, by definition, an element of A. Therefore, Definition 3.1.4 shows that we must have that $A = A$.

Symmetric: Let A, B be sets where every element of A is also an element of B and every element of B is also an element of A. Then, by Definition 3.1.4 we have that $A = B$ and $B = A$.

Transitive: Let A, B, C be sets where we have that $A = B$ and $B = C$. Then, since by Definition 3.1.4, every element of A is an element of B and every element of B is an element of C. Therefore, every element of A must also be an element of C and vice versa. Therefore, if $A = B$ and $B = C$ we must have that $A = C$. □

3.1.2. Using only Definition 3.1.4, Axiom 3.1, Axiom 3.2, and Axiom 3.3, prove that the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct (i.e., no two of them are equal to each other.)

Proof. Axiom 3.2 states that $\emptyset$ doesn't contain any elements. We can see that $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ do contain elements and therefore, by Definition 3.1.4 (we will emit similar justifications for the rest of the proof), we see that $\emptyset$ is not equal to any of them.

$\{\emptyset\}$ only contains $\emptyset$. $\{\{\emptyset\}\}$ does not contain $\emptyset$ so they are not equal. $\{\emptyset, \{\emptyset\}\}$ contains the element $\emptyset$ while $\{\emptyset\}$ does not. Therefore $\{\emptyset\}$ is not equal to the remaining sets.

$\{\{\emptyset\}\}$ only contains $\{\emptyset\}$ while $\{\emptyset, \{\emptyset\}\}$ also contains $\emptyset$. Therefore, $\{\{\emptyset\}\}$ is not equal to the remaining sets. □

3.1.3. Prove the remaining claims in Lemma 3.1.13.

Proof. If $x \in A \cup B$ then either $x \in A$ or $x \in B$, or both. Therefore, $x \in B \cup A$ since it was shown to either be an element of B or A or both. The converse argument is the same. Thus, the union of sets is commutative.

If $x \in A \cup A$ then $x \in A$ and conversely if $x \in A$ then $x \in A \cup A$. Therefore, $A \cup A = A$. If $x \in A \cup \emptyset$ then $x \in A$ since $\emptyset$ doesn't contain any elements. Conversely, if $x \in A$ then $x \in A \cup \emptyset$ since $\emptyset$ doesn't contain any elements. Therefore, $A \cup \emptyset = A$. Similar arguments hold for $\emptyset \cup A$ showing that $\emptyset \cup A = A$. □

3.1.4. Prove the remaining claims in Proposition 3.1.18.

Proof. The remaining claims in Proposition 3.1.18 are:

1. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

2. If $A \subset B$ and $B \subset C$ then $A \subset C$ (Note that in LaTeX a proper subset is denoted with $\subset$).

If $A \subseteq B$ and $B \subseteq A$, then if $x \in A$ we must have that $x \in B$ and vice versa. Therefore, from Definition 3.1.14 we have that $A = B$.

If $A \subset B$ and $B \subset C$, then if $x \in A$ we must have that $x \in B$ and furthermore if $x \in B$ we must have that $x \in C$. Thus, $A \subset C$. □

3.1.5. Let A, B be sets. Show that the three statements $A \subseteq B$, $A \cup B = B$, $A \cap B = A$ are logically equivalent (any one of them implies the other two).

Proof. Suppose $A \subseteq B$. We will show this implies the other two statements.

[$A \cup B = B$]: If $x \in B$ then by definition of union $x \in A \cup B$. Conversely, if $x \in A \cup B$, we must have that $x \in A$ or $x \in B$. If $x \in A$, we must have that $x \in B$ since $A \subseteq B$. Therefore, if $A \subseteq B$, then $A \cup B = B$.

[$A \cap B = A$]: If $x \in A \cap B$ then $x \in A$ and $x \in B$. Since $A \subseteq B$ we must have that $x \in A$. Conversely, if $x \in A$ we must have that $x \in B$, thus $x \in A \cap B$, since $A \subseteq B$. Therefore, if $A \subseteq B$, then $A \cap B = A$. □

3.1.6. Prove Proposition 3.1.28. (Hint: one can use some of these claims to prove others. Some of the claims have also appeared previously in Lemma 3.1.13.)

Proof. We will prove the laws of Boolean algebra. Recall:

Let A, B, C be sets, and let X be a set containing A, B, C as subsets.

(a) (*Minimal element*) *We have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.*

We already proved $A \cup \emptyset = A$ in Exercise 3.1.3. From Definition 3.1.23 the intersection of any set with the empty set must be the empty set. Therefore, $A \cap \emptyset = \emptyset$.

(b) (*Maximal element*) *We have $A \cup X = X$ and $A \cap X = A$.*

If $x \in A \cup X$ then $x \in A$ or $x \in X$ or both. If $x \in A$, then since $A \subseteq X$, we must have that $x \in X$ either way and therefore $A \cup X = X$. Since $A \subseteq X$, we have already seen this being equivalent with $A \cap X = A$ from Exercise 3.1.5.

(c) (*Identity*) *We have $A \cap A = A$ and $A \cup A = A$.*

If $x \in A \cap A$ then $x \in A$ and therefore $A \cap A = A$. If $x \in A \cup A$ then $x \in A$ and therefore $A \cup A = A$.

(d) (*Commutativity*) *We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$.*

The commutativity of the union was proved in Exercise 3.1.3 for Lemma 3.1.13. If $x \in A \cap B$ then $x \in A$ and $x \in B$. Therefore, we see that $x \in B \cap A$ so that $A \cap B = B \cap A$.

(e) (*Associativity*) *We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.*

The text proved the associativity of the union in Lemma 3.1.13. If $x \in (A \cap B) \cap C$ then $x \in (A \cap B)$ and $x \in C$. Since $x \in (A \cap B)$, x is also an element of A and B. Thus, x is an element of the sets A, B, C and with Definition 3.1.23 we see that $x \in A \cap (B \cap C)$ and the desired equality follows.

(f) (*Distributivity*) *We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*

If $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in (B \cup C)$ and hence $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$, or both. Therefore $x \in (A \cap B) \cup (A \cap C)$. Conversely, if $x \in (A \cap B) \cup (A \cap C)$ then $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$, or both. Hence $x \in A$ and $x \in (B \cup C)$ and therefore $x \in A \cap (B \cup C)$. We conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

If $x \in A \cup (B \cap C)$ then $x \in A$ or $x \in B$ and $x \in C$. Hence $x \in A$ or $x \in B$, and, $x \in A$ or $x \in C$. Therefore $x \in (A \cup B) \cap (A \cup C)$. Conversely, if $x \in (A \cup B) \cap (A \cup C)$ then $x \in A$ or $x \in B$, and, $x \in A$ or $x \in C$. Hence $x \in A$ or $x \in B$ and $x \in C$ and therefore $x \in A \cup (B \cap C)$. We conclude that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(g) (Partition) We have $A \cup (X \setminus A) = X$ *and* $A \cap (X \setminus A) = \emptyset$.

The union of A and $(X \setminus A)$ is all of X. Thus, $A \cup (X \setminus A) = X$.

Since A and $(X \setminus A)$ are disjoint sets (i.e., they have no elements in common) we conclude that $A \cap (X \setminus A) = \emptyset$.

(h) (De Morgan laws) We have $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ *and* $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

If $x \in X \setminus (A \cup B)$ then $x \notin A \cup B$ and hence $x \notin A$ and $x \notin B$. Thus, $x \in (X \setminus A)$ and $x \in (X \setminus B)$ and therefore $x \in (X \setminus A) \cap (X \setminus B)$. Conversely, if $x \in (X \setminus A) \cap (X \setminus B)$ then $x \in (X \setminus A)$ and $x \in (X \setminus B)$ and hence $x \notin A$ and $x \notin B$. Thus, $x \notin A \cup B$ and therefore $x \in X \setminus (A \cup B)$. We conclude that $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$.

If $x \in X \setminus (A \cap B)$ then $x \notin A \cap B$ and hence $x \notin A$ or $x \notin B$. Thus, $x \in (X \setminus A)$ or $x \in (X \setminus B)$ and therefore $x \in (X \setminus A) \cup (X \setminus B)$. Conversely, if $x \in (X \setminus A) \cup (X \setminus B)$ then $x \in (X \setminus A)$ or $x \in (X \setminus B)$ and hence $x \notin A$ or $x \notin B$. Thus, $x \notin A \cap B$ and therefore $x \in X \setminus (A \cap B)$. We conclude that $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

□

3.1.7. Let A, B, C be sets. Show that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Furthermore, show that $C \subseteq A$ and $C \subseteq B$ if and only if $C \subseteq A \cap B$. In a similar spirit, show that $A \subseteq A \cup B$ and $B \subseteq A \cup B$, and furthermore that $A \subseteq C$ and $B \subseteq C$ if and only if $A \cup B \subseteq C$.

Proof. If $x \in A \cap B$ then $x \in A$ and therefore $A \cap B \subseteq A$. Similar argument holds for $A \cap B \subseteq B$.

If $C \subseteq A$ and $C \subseteq B$ then if $x \in C$ we must have that $x \in A$ and $x \in B$ and therefore $x \in A \cap B$. Thus, $C \subseteq A \cap B$. Conversely, if $C \subseteq A \cap B$ then if $x \in C$ we must have that $x \in A \cap B$ and therefore $x \in A$ and $x \in B$. Thus, $C \subseteq A$ and $C \subseteq B$.

If $x \in A$ then $x \in A \cup B$ by definition of union of sets and therefore $A \subseteq A \cup B$. Same argument holds for $B \subseteq A \cup B$.

If $A \subseteq C$ and $B \subseteq C$ then if $x \in A$ we must have that $x \in A \cup B$ from the previous deduction above. Since $B \subseteq C$ we must have that $x \in C$ and therefore $A \cup B \subseteq C$. A similar argument holds when assuming $x \in B$. Conversely, if $A \cup B \subseteq C$, we know from the previous deduction above that $A \subseteq A \cup B$ and $B \subseteq A \cup B$ and therefore we have that $A \subseteq C$ and $B \subseteq C$. □

3.1.8. Let A, B be sets. Prove the *absorption laws* $A \cap (A \cap B) = A$ and $A \cup (A \cap B) = A$.

Proof. Recall from the previous exercise that $A \cap B \subseteq A$. The set B may not be a subset of A but $A \cap B$ is a subset of A. Furthermore, from the definition of intersection, the intersection of a set with one of its subsets is equal to that subset. Therefore if $x \in A \cap (A \cap B)$ we must have that $x \in A$ and vice versa. Hence, $A \cap (A \cap B) = A$.

If $x \in A \cup (A \cup B)$ then $x \in A$ or $x \in (A \cup B)$. If $x \in (A \cup B)$ then $x \in A$ or $x \in B$. Remember, this is *inclusive* or, and therefore, either way we have that $x \in A$. The converse direction has a similar argument. Hence, $A \cup (A \cup B) = A$. □

3.1.9. Let A, B, X be sets such that $A \cup B = X$ and $A \cap B = \emptyset$. Show that $A = X \setminus B$ and $B = X \setminus A$.

Proof. Recall from De Morgan's laws that $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$.

Since $A \cup B = X$ and $A \cup (X \setminus A) = X$, we see that by the substitution rule that $B = (X \setminus A)$.

Since the set A in De Morgan's laws above was arbitrary, we can replace A with B to get that $A = (X \setminus B)$. □

3.1.10. Let A and B be sets. Show that the three sets $A \setminus B$, $A \cap B$, and $B \setminus A$ are disjoint, and that their union is $A \cup B$.

Proof. The intersection of disjoint sets is $\emptyset$.

Note that the set $A \setminus B$ is disjoint from B since it is the set A minus any elements that are in B, that is $(A \setminus B) \cap B = \emptyset$. There is a similar argument for $B \setminus A$ with roles reversed.

We then have that $(A \setminus B) \cap (B \setminus A) = \emptyset$ by definition of intersection since the first set doesn't contain any elements of B while the second set doesn't contain any elements of A.

Additionally, $(A \setminus B) \cap (A \cap B) = \emptyset$ and $(B \setminus A) \cap (A \cap B) = \emptyset$, as the intersections of these sets with $A \cap B$ will be empty in both cases as $(A \setminus B)$ and $(B \setminus A)$ have empty intersections with B and A, respectively. □

3.1.11. Show that the axiom of replacement implies the axiom of specification.

Proof. Let A be a set.

We can derive the axiom of specification from the axiom of replacement by noticing that if we set the property $P(x, y)$ to be the property that is true when both $y = x$ and the property $P(x)$ is true, we will have:

$$\begin{aligned} \{y \mid P(x,y) \textit{ is true for some } x \in A\} &\implies \{y = x \in A \mid P(x, y = x) \textit{ is true}\} \\ &\implies \{x \in A \mid P(x) \textit{ is true}\} \end{aligned}$$

which then gives us the axiom of specification. □

§3.2 Russell's paradox

3.2.1. Show that the universal specification axiom, Axiom 3.8, if assumed to be true, would imply Axioms 3.2, 3.3, 3.4, 3.5, and 3.6. (If we assume that all natural numbers are objects, we also obtain Axiom 3.7.) Thus, this axiom, if permitted, would simplify the foundations of set theory tremendously (and can be viewed as one basis for an intuitive model of set theory known as "naive set theory"). Unfortunately, as we have seen, Axiom 3.8 is "too good to be true"!

Proof. With the universal specification axiom, for any property $P(x)$ pertaining to x there would exist a set $\{x \mid P(x) \textit{ is true}\}$. Something to keep in mind is that x is an arbitrary *object* and sets are considered objects.

Axiom 3.2: The existence of $\emptyset$.

For $\emptyset$, let $P(x)$ be the statement

$$P(x) \iff \text{"}x \in \mathbf{N} \text{ and } x = x{+}{+}\text{"}$$

which is trivially true by the Peano axioms. This constructs $\emptyset$.

Axiom 3.3: Singleton sets and pair sets.

For singleton sets, let $P(x)$ be the statement

$$P(x) \iff \text{"}x \text{ is the only element in this set"}$$

This would construct $\{x \mid x \text{ is the only element in this set}\} = \{x\}$.

For pair sets, let a, b be objects and $P(x)$ be the statement

$$P(x) \iff \text{"}x = a \text{ or } x = b\text{"}$$

This would construct $\{x \mid x = a \text{ or } x = b\} = \{a, b\}$.

Axiom 3.4: Pairwise union.

For sets A, B, let $P(x)$ be the statement

$$P(x) \iff \text{"}x \in A \text{ or } x \in B\text{"}$$

This would construct $\{x \mid x \in A \text{ or } x \in B\} = A \cup B$.

Axiom 3.5: Axiom of specification.

Let A be a set, let $P(x)$ be the statement

$$P(x) \iff \text{"}x \in A \text{ and } P(x) \text{ is true"}$$

This would construct $\{x \mid x \in A \text{ and } P(x) \text{ is true}\}$. This is the definition of Axiom 3.5.

Axiom 3.6: Replacement.

Let A be a set, let $P(x)$ be the statement

$$P(x) \iff \text{"}P(x, y) \text{ is true for some } y \in A\}\text{"}$$

This would construct $\{x \mid P(x, y) \text{ is true for some } y \in A\}$. This is the definition of Axiom 3.6. □

3.2.2. Use the axiom of regularity (and the singleton set axiom) to show that if A is a set, then $A \notin A$. Furthermore, show that if A and B are two sets, then either $A \notin B$ or $B \notin A$ (or both).

Proof. Let A be a set. If $A = \emptyset$ then it obviously doesn't contain itself. Thus, let us suppose that A is a non-empty set. From the the axiom of regularity A must contain at least one element which is either not a set, or is disjoint from A. Let us denote this element as x. From the singleton set axiom, if $A = \{x\}$ then it obviously doesn't contain A as x is either not a set, or is disjoint from A. If A has more elements, at the very least it would need to contain x and therefore x would need to be part of $A \in A$ if it were to exists. Let us see if we can construct some set A from singleton sets of x such that $A \in A$. Now, even if these other elements are themselves singleton sets of x, such as $A = \{x, \{x\}\}$, this still shows that $A \notin A$. This pattern can be continued indefinitely, i.e., $A = \{x, \{x, \{x, \{x \dots\}\}\}\}$ in an infinite regression and still $A \notin A$. Therefore, with the axiom of regularity (and the singleton set axiom) if A is a set, then $A \notin A$.

A possibly simpler proof of this is the following. Let A be a set and from the singleton set axiom we know that $\{A\}$ exists. However, from the axiom of regularity we know that there must be an element of $\{A\}$ which is disjoint from $\{A\}$. The only element of $\{A\}$ is A, thus A is disjoint from $\{A\}$. Therefore, $A \cap \{A\} = \emptyset$ and hence $A \notin A$.

Furthermore, if A and B are two sets, then we have four cases.

1. $A \cap B = \emptyset$. Since A and B are disjoint we must have that $A \notin B$ and $B \notin A$.
2. $A \subseteq B$. Since A is a subset of B we know that $B \notin A$ (if they are equal we still know that $A \notin A$ from above).
3. $B \subseteq A$. Since B is a subset of A we know that $A \notin B$ (if they are equal we still know that $A \notin A$ from above).
4. $A \cap B \neq \emptyset$. Since the intersection is non-zero and one set is not contained in the other (case (2) and (3) above), we must have that A and B share some elements but obviously we still have that $A \notin B$ and $B \notin A$.

Therefore, if A and B are two sets, then either $A \notin B$ or $B \notin A$ (or both). □

3.2.3. Show (assuming the other axioms of set theory) that the universal specification axiom, Axiom 3.8, is equivalent to an axiom postulating the existence of a "universal set" Ω consisting of all objects (i.e., for all objects x, we have $x \in \Omega$). In other words, if Axiom 3.8 is true, then a universal set exists, and conversely, if a universal set exists, then Axiom 3.8 is true. (This may explain why Axiom 3.8 is called the axiom of universal specification). Note that if a universal set Ω existed, then we would have $\Omega \in \Omega$ by Axiom 3.1, contradicting Exercise 3.2.2. Thus the axiom of foundation specifically rules out the axiom of universal specification.

Proof. Axiom 3.8 is equivalent to an axiom postulating the existence of a "universal set" Ω because if we let $P(x)$ be the statement that

$$P(x) \iff \text{"}x \text{ is an object"}$$

this would construct a set $\{x \mid P(x) \text{ is true}\} = \{x \mid x \text{ is an object}\} = \Omega$.

An argument in the reverse direction shows that if a universal set Ω exists, that it too implies Axiom 3.8. □

§3.3 Functions

3.3.1. Show that the definition of equality in Definition 3.3.7 is reflexive, symmetric, and transitive. Also verify the substitution property: if $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$ are functions such that $f = \tilde{f}$ and $g = \tilde{g}$, then $g \circ f = \tilde{g} \circ \tilde{f}$.

Proof. Let $f : X \to Y$, $g : X \to Y$, and $h : X \to Y$.

Reflexive: f has the same domain and range as itself and for all $x \in X$ we must have $f(x) = f(x)$ from Definition 3.3.1 — via the *vertical line test*. Thus, $f = f$.

Symmetric: f and g have the same domain and range and if $f(x) = g(x)$ for all $x \in X$ then obviously $g(x) = f(x)$ for all $x \in X$ as well. Thus, $f = g$ and $g = f$.

Transitive: We saw that $f = g$. Now suppose that by the same reasoning we have that $g = h$. Then f and h must have the same domain and range and for all $x \in X$ we must have that $f(x) = h(x)$ from the fact that we had $f(x) = g(x)$ and $g(x) = h(x)$. Thus, if $f = g$ and $g = h$, then $f = h$.

$(g \circ f = \tilde{g} \circ \tilde{f})$:

$$\begin{aligned}
(g \circ f)(x) &= g \circ (f(x)) \\
&= g \circ (\tilde{f}(x)) && [f(x) = \tilde{f}(x)] \\
&= g(\tilde{f}(x)) \\
&= \tilde{g}(\tilde{f}(x)) && [g(\tilde{f}(x)) = \tilde{g}(\tilde{f}(x))] \\
&= \tilde{g} \circ (\tilde{f}(x)) \\
&= (\tilde{g} \circ \tilde{f})(x)
\end{aligned}$$

□

3.3.2. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are both injective, then so is $g \circ f$; similarly, show that if f and g are both surjective, then so is $g \circ f$.

Proof. If f and g are both injective then for each $x \in X$ we must have a unique $y = f(x) \in Y$. The same reasoning applies to each $y \in Y$ and that we must have a unique $z = g(y) = g(f(x))$. Therefore, $g \circ f$ is injective.

If f and g are both surjective then for each $y \in Y$ there exists an x such that $y = f(x)$. The same reasoning applies for each $z \in Z$ and that there exists a y such that $z = g(y) = g(f(x))$. Therefore, $g \circ f$ is surjective. □

3.3.3. When is the empty function injective? surjective? bijective?

Recall that the empty function is $f : \emptyset \to X$.

The empty function is always injective. The empty function is surjective when $X = \emptyset$. The empty function is bijective when $f : \emptyset \to \emptyset$.

3.3.4. In this section we give some cancellation laws for composition. Let $f : X \to Y$, $\tilde{f} : X \to Y$, $g : Y \to Z$, and $\tilde{g} : Y \to Z$ be functions. Show that if $g \circ f = g \circ \tilde{f}$ and g is injective, then $f = \tilde{f}$. Is the same statement true if g is not injective? Show that if $g \circ f = \tilde{g} \circ f$ and f is surjective, then $g = \tilde{g}$. Is the same statement true if f is not surjective?

Proof. If $g \circ f = g \circ \tilde{f}$ and g is injective, then $g(f(x)) = g(\tilde{f}(x))$ and $f(x) = \tilde{f}(x)$ for all $x \in X$ (g is injective) and therefore we must have that $f = \tilde{f}$ by Definition 3.3.7.

The same statement is false if g is not injective.

If $g \circ f = \tilde{g} \circ f$ and f is surjective, then $g(f(x)) = \tilde{g}(f(x))$ and since f is surjective we know that $g(f(x)) = \tilde{g}(f(x))$ is true for the entire domain Y and therefore $g = \tilde{g}$ by Definition 3.3.7.

The same statement is false if f is not surjective. □

3.3.5. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if $g \circ f$ is injective, then f must be injective. Is it true that g must also be injective? Show that if $g \circ f$ is surjective, then g must be surjective. Is it true that f must also be surjective?

Proof. If $g \circ f$ is injective, then we know that

$$g(f(x)) = g(f((x')) \implies x = x'.$$

Let us suppose that f is not injective. Then there would be at least two different elements of X such that they would map to the same element in Y, say $x \neq x'$ mapping to $f(x) = f(x')$. If we compose this with g then we will arrive at $g(f(x)) = g(f(x'))$. But this form implies that $x = x'$, a contradiction. Therefore, f is injective. g need not be injective.

If $g \circ f$ is surjective, then we know that for every $z \in Z$, there exists $x \in X$ such that $z = g(f(x))$. Furthermore, this shows that for every $z \in Z$ that there exists $y = f(x) \in Y$ such that $z = g(f(x))$ and therefore g is surjective. f need not be surjective. □

3.3.6. Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse. Verify the cancellation laws $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$. Conclude that f^{-1} is also invertible, and has f as its inverse (thus $(f^{-1})^{-1} = f$).

Proof.

$[f^{-1}(f(x)) = x]$:

$$\begin{aligned} f(x) &= y && \text{[definition of } f\text{]} \\ f^{-1}(f(x)) &= f^{-1}(y) && \text{[applying } f^{-1} \text{ to both sides]} \\ f^{-1}(f(x)) &= x && [f^{-1}(y) = x] \end{aligned}$$

$(f(f^{-1}(y)) = y)$:

$$\begin{aligned} f^{-1}(y) &= x && \text{[definition of } f^{-1}\text{]} \\ f(f^{-1}(x)) &= f(x) && \text{[applying } f \text{ to both sides]} \\ f(f^{-1}(x)) &= y && [f(x) = y] \end{aligned}$$

$(f^{-1})^{-1} = f)$: For us to conclude that f^{-1} is invertible, we need to prove it is also a bijection.

(injective): Suppose that $x = x'$ where $x = f^{-1}(y)$ and $x' = f^{-1}(y')$ (this can be done since f is bijective) so that we have that $f^{-1}(y) = f^{-1}(y')$. Then,

$$\begin{aligned} f^{-1}(y) &= f^{-1}(y') \\ f(f^{-1}(y)) &= f(f^{-1}(y')) && \text{[applying } f \text{ to both sides]} \\ y &= y' \end{aligned}$$

Showing that f^{-1} is injective.

(surjective): For f^{-1} to be surjective we need to show that for every $x \in X$ there exists $y \in Y$ such that $x = f^{-1}(y)$. Then

$$\begin{aligned} x &= f^{-1}(y) && [f \text{ is bijective}] \\ f(x) &= f(f^{-1}(y)) && \text{[applying } f \text{ to both sides]} \\ f(x) &= y \end{aligned}$$

Showing that for every $x \in X$ there exists $y \in Y$ such that $x = f^{-1}(y)$. Thus, f^{-1} is surjective.

We conclude that f^{-1} is invertible. The value of y is denoted by $(f^{-1})^{-1}(x)$; but we also have that $y = f(x)$ and therefore deduce that $(f^{-1})^{-1} = f$. □

3.3.7. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are bijective, then so is $g \circ f$, and we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof.

(injective): Suppose that $(g \circ f)(x) = (g \circ f)(x')$. Then,

$$\begin{aligned} (g \circ f)(x) &= (g \circ f)(x') \\ g \circ f(x) &= g \circ f(x') \\ g(f(x)) &= g(f(x')) \\ g^{-1}(g(f(x))) &= g^{-1}(g(f(x'))) && \text{[inverse function exists for bijection]} \\ f(x) &= f(x') \\ f^{-1}(f(x)) &= f^{-1}(f(x')) && \text{[inverse function exists for bijection]} \\ x &= x' \end{aligned}$$

(surjective): Let $z = (g \circ f)(x)$. Then,

$$\begin{aligned} &(g \circ f)(x) \\ &g \circ f(x) \\ &g(f(x)) \\ &g^{-1}(g(f(x))) && \text{[inverse function exists for bijection]} \\ &f(x) \\ &f^{-1}(f(x)) && \text{[inverse function exists for bijection]} \\ &x \end{aligned}$$

Thus, $g \circ f$ is bijective.

Seeing that $(g \circ f)^{-1}(g \circ f)(x) = x$ we can find what the inverse would be by showing that what it takes to arrive at x from $(g \circ f)(x)$:

$$\begin{aligned}
&(g \circ f)(x) \\
&g(f(x)) \\
&g^{-1}(g(f(x))) && \text{[applied } g^{-1} \text{ first]} \\
&f(x) \\
&f^{-1}(f(x)) && \text{[applied } f^{-1} \text{ second]} \\
&x
\end{aligned}$$

Since function composition is applied right from left we see that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

3.3.8. If X is a subset of Y, let $\iota_{X \to Y} : X \to Y$ be the inclusion map from X to Y, defined by mapping $x \mapsto x$ for all $x \in X$, i.e., $\iota_{X \to Y}(x) := x$ for all $x \in X$. The map $\iota_{X \to X}$ is in particular called the identity map on X.

(a) Show that if $X \subseteq Y \subseteq Z$ then $\iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z}$.

Proof. To show that the two functions are equal, i.e. $\iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z}$, we note that they both have the same domain and range, X and Z respectively. Additionally, for any $x \in X$ since the individual functions' outputs are just the inputs, we must have that $(\iota_{Y \to Z} \circ \iota_{X \to Y})(x) = x$ and $(\iota_{X \to Z})(x) = x$, showing that they are equal. □

(b) Show that if $f : A \to B$ is any function, then $f = f \circ \iota_{A \to A} = \iota_{B \to B} \circ f$.

Proof. f and $f \circ \iota_{A \to A}$ and $\iota_{B \to B} \circ f$ all have the same domain and range, A and B, respectively. Let $a \in A$. Then,

$$\begin{aligned}
f(a) &= (f \circ \iota_{A \to A})(a) \\
&= f(\iota_{A \to A}(a)) \\
&= f(a) && [\iota_{A \to A}(a) = a]
\end{aligned}$$

showing that $f = f \circ \iota_{A \to A}$. Continuing,

$$\begin{aligned}
f(a) &= (\iota_{B \to B} \circ)(a) \\
&= \iota_{B \to B}(f(a)) \\
&= f(a) && [\iota_{B \to B}(b) = b \text{ where } b = f(a)]
\end{aligned}$$

showing that $f = \iota_{B \to B} \circ f$. Therefore, if $f : A \to B$ is any function, then $f = f \circ \iota_{A \to A} = \iota_{B \to B} \circ f$. □

(c) Show that, if $f : A \to B$ is a bijective function, then $f \circ f^{-1} = \iota_{B \to B}$ and $f^{-1} \circ f = \iota_{A \to A}$.

Proof. Since f is bijective we know that it has an inverse, f^{-1} and also that $b = f(a)$ and $a = f^{-1}(b)$ both exist and are unique.

$[f \circ f^{-1} = \iota_{B \to B}]$:

Both these functions have the same domain and range, which is the set B. We know that $\iota_{B \to B}(b) = b$ and $(f \circ f^{-1})(b)$ gives us,

$$\begin{aligned}
(f \circ f^{-1})(b) &= f(f^{-1}(b)) \\
&= f(a) && [a = f^{-1}(b)] \\
&= b && [b = f(a)]
\end{aligned}$$

Therefore, $f \circ f^{-1} = \iota_{B \to B}$.

$[f^{-1} \circ f = \iota_{A \to A}]$:

Both functions have the same domain and range, which is the set A. We know that $\iota_{A \to A}(a) = a$ and $f^{-1} \circ f = \iota_{A \to A}$ gives us,

$$\begin{aligned}
(f^{-1} \circ f)(a) &= f^{-1}(f(a)) \\
&= f^{-1}(b) && [b = f(a)] \\
&= a && [a = f^{-1}(b)]
\end{aligned}$$

Therefore, $f^{-1} \circ f = \iota_{A \to A}$. □

(d) Show that if X and Y are disjoint sets, and $f : X \to Z$ and $g : Y \to Z$ are functions, then there is a unique function $h : X \cup Y \to Z$ such that $h \circ \iota_{X \to X \cup Y} = f$ and $h \circ \iota_{Y \to X \cup Y} = g$.

Proof. Let $a \in X \cup Y$ and let us define the function h to be

$$h(a) = \begin{cases} f(a), & \text{if } a \in X \\ g(a), & \text{if } a \in Y \end{cases}$$

Then, from the solutions above it is easy to see that $h \circ \iota_{X \to X \cup Y} = f$ and $h \circ \iota_{Y \to X \cup Y} = g$. □

§3.4 Images and inverse images

3.4.1. Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse. Let V be any subset of Y. Prove that the forward image of V under f^{-1} is the same set as the inverse image of V under f; thus the fact that both sets are denoted by $f^{-1}(V)$ will not lead to any inconsistency.

Proof. The forward image of V under f^{-1} is the set

$$f^{-1}(V) = \{f^{-1}(y) \mid y \in V\}$$

and the inverse image of V under f is the set

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$$

These sets are equivalent because f is bijective and therefore $x = f^{-1}(y)$ and $y = f(x)$ are unique. □

3.4.2. Let $f : X \to Y$ be a function from one set X to another set Y, let S be a subset of X, and let U be a subset of Y. What, in general, can one say about $f^{-1}(f(S))$ and S? What about $f\left(f^{-1}(U)\right)$ and U?

In general, $f^{-1}(f(S))$ and S are not equal as we saw in Examples 3.4.5 and 3.4.6. However, if f is bijective they will be equal.

In general, $f\left(f^{-1}(U)\right)$ and U are not equal as we saw in Examples 3.4.5 and 3.4.6. However, if f is bijective they will be equal.

3.4.3. Let A, B be two subsets of a set X, and let $f : X \to Y$ be a function. Show that $f(A \cap B) \subseteq f(A) \cap f(B)$, that $f(A) \setminus f(B) \subseteq f(A \setminus B)$, $f(A \cup B) = f(A) \cup f(B)$. For the first two statements, is it true that the $\subseteq$ relation can be improved to $=$?

Proof. Let

$$\begin{aligned}
f(A) &= \{f(x) \mid x \in A\} \\
f(B) &= \{f(x) \mid x \in B\} \\
f(A \cup B) &= \{f(x) \mid x \in A \cup B\} \\
f(A \cap B) &= \{f(x) \mid x \in A \cap B\} \\
f(A) \cup f(B) &= \{f(x) \mid f(x) \in f(A) \text{ or } f(x) \in f(B)\} \\
f(A) \cap f(B) &= \{f(x) \mid f(x) \in f(A) \text{ and } f(x) \in f(B)\} \\
f(A \setminus B) &= \{f(x) \mid x \in A \setminus B\} \\
f(A) \setminus f(B) &= \{f(x) \mid f(x) \in f(A) \text{ and } f(x) \notin f(B)\}.
\end{aligned}$$

[$f(A \cap B) \subseteq f(A) \cap f(B)$]: If $y \in f(A \cap B)$ then $y = f(x)$ such that $x \in A$ and $x \in B$ and hence $y \in f(A)$ and $y \in f(B)$. Therefore, $f(A \cap B) \subseteq f(A) \cap f(B)$.

[$f(A) \setminus f(B) \subseteq f(A \setminus B)$]: If $y \in f(A) \setminus f(B)$ then $y \in f(A)$ and $y \notin f(B)$ and hence $y = f(x)$ such that $x \in A$ and $x \notin B$. That is, $x \in A \setminus B$ which means that $y \in f(A \setminus B)$. Therefore, $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

[$f(A \cup B) = f(A) \cup f(B)$]: If $y \in f(A \cup B)$ then $y = f(x)$ such that $x \in A$ or $x \in B$ and hence $y \in f(A)$ or $y \in f(B)$. Therefore, $f(A \cup B) \subseteq f(A) \cup f(B)$. Conversely, if $y \in f(A) \cup f(B)$ then $y = f(x)$ such that $x \in A$ or $x \in B$ and hence $y \in f(A \cup B)$. Therefore, $f(A) \cup f(B) \subseteq f(A \cup B)$. We conclude that $f(A \cup B) = f(A) \cup f(B)$. ☐

Yes, the in the first two statements the $\subseteq$ relation can be improved to $=$.

3.4.4. Let $f : X \to Y$ be a function from one set X to another set Y, and let U, V be subsets of Y. Show that $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$, that $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$, and that $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$.

Proof. Let

$$\begin{aligned}
f^{-1}(U) &= \{x \in X \mid f(x) \in U\} \\
f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\
f^{-1}(U \cup V) &= \{x \in X \mid f(x) \in U \cup V\} \\
f^{-1}(U \cap V) &= \{x \in X \mid f(x) \in U \cap V\} \\
f^{-1}(U) \cup f^{-1}(V) &= \{x \in X \mid x \in f^{-1}(U) \text{ or } x \in f^{-1}(V)\} \\
f^{-1}(U) \cap f^{-1}(V) &= \{x \in X \mid x \in f^{-1}(U) \text{ and } x \in f^{-1}(V)\} \\
f^{-1}(U \setminus V) &= \{x \in X \mid f(x) \in U \setminus V\} \\
f^{-1}(U) \setminus f^{-1}(V) &= \{x \in X \mid x \in f^{-1}(U) \text{ and } x \notin f^{-1}(V)\}.
\end{aligned}$$

[$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$]: If $x \in f^{-1}(U \cup V)$ then $x = f^{-1}(y)$ such that $f(x) \in U \cup V$ and thus $f(x) \in U$ or $f(x) \in V$, hence $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$. Thus, $f^{-1}(U \cup V) \subseteq f^{-1}(U) \cup f^{-1}(V)$. Conversely, if $x \in f^{-1}(U) \cup f^{-1}(V)$ then $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$ such that $f(x) \in U$ or $f(x) \in V$ and thus $f(x) \in U \cup V$, hence $x \in f^{-1}(U \cup V)$. Thus, $x \in f^{-1}(U) \cup f^{-1}(V) \subseteq f^{-1}(U \cup V)$. Therefore, $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.

[$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$]: If $x \in f^{-1}(U \cap V)$ then $x = f^{-1}(y)$ such that $f(x) \in U \cap V$ and thus $f(x) \in U$ and $f(x) \in V$, hence $x \in f^{-1}(U)$ and $x \in f^{-1}(V)$. Thus, $f^{-1}(U \cap V) \subseteq f^{-1}(U) \cap f^{-1}(V)$. Conversely, if $x \in f^{-1}(U) \cap f^{-1}(V)$ then $x \in f^{-1}(U)$ and $x \in f^{-1}(V)$ such that $f(x) \in U$ and $f(x) \in V$

and thus $f(x) \in U \cap V$, hence $f^{-1}(U \cap V)$. Thus, $x \in f^{-1}(U) \cap f^{-1}(V) \subseteq f^{-1}(U \cap V)$. Therefore, $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$.

$[f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)]$: If $x \in f^{-1}(U \setminus V)$ then $x = f^{-1}(y)$ such that $f(x) \in U \setminus V$ and thus $f(x) \in U$ and $f(x) \notin V$, hence $x \in f^{-1}(U)$ and $x \notin f^{-1}(V)$. Thus, $f^{-1}(U \setminus V) \subseteq f^{-1}(U) \setminus f^{-1}(V)$. Conversely, if $x \in f^{-1}(U) \setminus f^{-1}(V)$ then $x \in f^{-1}(U)$ and $x \notin f^{-1}(V)$ such that $f(x) \in U$ and $f(x) \notin V$ and thus $f(x) \in U \setminus V$, hence $x \in f^{-1}(U \setminus V)$. Thus, $x \in f^{-1}(U) \setminus f^{-1}(V) \subseteq f^{-1}(U \setminus V)$. Therefore, $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$. □

3.4.5. Let $f : X \to Y$ be a function from one set X to another set Y. Show that $f\left(f^{-1}(S)\right) = S$ for every $S \subseteq Y$ if and only if f is surjective. Show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective.

Proof. If f is surjective, then by Definition 3.3.17, for every $S \subseteq Y$ there must exist $f^{-1}(S) \subseteq X$ such that $f(f^{-1}(S)) = S$. Conversely, if $f(f^{-1}(S)) = S$ then we know that for any $S \subseteq Y$ there exists $f^{-1}(S)$ such that $f(f^{-1}(S)) = S$ and by Definition 3.3.17, f is surjective. Therefore, $f\left(f^{-1}(S)\right) = S$ for every $S \subseteq Y$ if and only if f is surjective.

If f is injective then f will map every element of S to a unique element of $f(S)$. The inverse image map will then map each unique element of $f(S)$ back to its original element in S so that we have that $f^{-1}(f(S)) = S$. Conversely, if $f^{-1}(f(S)) = S$ for any $S \subseteq X$. Now, suppose we have $f(S) = f(S')$ for $S \subseteq X$ and $S' \subseteq X$. Then,

$$\begin{aligned} f(S) &= f(S') \\ f^{-1}(f(S)) &= f^{-1}(f(S')) && \text{[applying } f^{-1} \text{ to both sides]} \\ S &= S' && \text{[by hypothesis]} \end{aligned}$$

Therefore, $f^{-1}(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective. □

3.4.6. Prove Lemma 3.4.9. (Hint: start with the set $\{0,1\}^X$ and apply the replacement axiom, replacing each function f with the object $f^{-1}(\{1\})$.) See also Exercise 3.5.11.

Proof. Lemma 3.4.9 claims: Let X be a set. Then $\{Y \mid Y \text{ is a subset of } X\}$ is a set.

To prove this we will follow the hint and start with the set $\{0,1\}^X$ and we will show that by using the replacement axiom that we will create a set that contains the subsets of the set X, thus proving the Lemma.

Let $A = \{0,1\}^X$ be the set of all functions with domain X and range Y. Let $P(f, S)$ be the statement that $S = f^{-1}(\{1\})$, where S is any subset of X. Observe that for each $f \in A$ there is at most one S for which $P(f, S)$ is true since S is the inverse image of $\{1\}$ under the map f.

Thus, using the axiom of replacement we know that the set $\{S \mid P(f, S) \text{ is true for some } f \in A\}$ exists and that this set is equal to $\{S \mid S \subseteq X\}$, thereby proving Lemma 3.4.9. □

3.4.7. Let X, Y be sets. Define a partial function from X to Y to be any function $f : X' \to Y'$ whose domain X' is a subset of X, and whose range Y' is a subset of Y. Show that the collection of all partial functions from X to Y is itself a set. (Hint: use Exercise 3.4.6, the power set axiom, the replacement axiom, and the union axiom.)

Proof. To show this, we construct two sets using the method of Exercise 3.4.6 above. One set is the subsets of X which we label 2^X and the other is the subsets of Y which we label 2^Y.

Let $P(X', Y', S)$ be the statement that $S = Y'^{X'}$ is equal to the power set of X' and Y', which we know exists by the power set axiom. Additionally, observe that for each X' and Y' there is at most one S for which $P(X', Y', S)$ is true since S is the power set of X' and Y'.

Then, using the axiom of replacement we know that the set

$$\{S \mid P(X', Y', S) \text{ is true for some } X' \in 2^X \text{ and for some } Y' \in 2^Y\}$$

exists. Let us denote this set A and note that this set is equal to $\{S = Y'^{X'} \mid X' \in 2^X \text{ and } Y' \in 2^Y\}$ and its elements are the sets of partial functions from X to Y. Using the union axiom we have:

$$\bigcup A$$

which is the collection of all partial functions from X to Y which itself is a set. ☐

3.4.8. Show that Axiom 3.4 can be deduced from Axiom 3.1, Axiom 3.3 and Axiom 3.11.

Proof. Axiom 3.4 is the pairwise union axiom which says that given any two sets A, B, there exists a set $A \cup B$.

Recall that Axiom 3.1 is the axiom that sets are themselves objects, Axiom 3.3 is the axiom that singleton sets and pair sets exist, and Axiom 3.11 is the union axiom.

Let x_1, x_2, y_1, y_2 be objects and using Axiom 3.3 create the sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. From Axiom 3.1, we know that X and Y are objects themselves, so we can use Axiom 3.3 again to create the set $A = \{X, Y\}$. Then, using Axiom 3.11 we take the union to get $\bigcup A = \{x_1, x_2, y_1, y_2\} = X \cup Y$, thereby deriving Axiom 3.4. ☐

3.4.9. Show that if β and β' are two elements of a set I, and to each $\alpha \in I$ we assign a set A_α, then $\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}$ and so the definition of $\bigcap_{\alpha \in I} A_\alpha$ defined in (3.3) does not depend on β. Also explain why (3.4) is true.

Proof. Since β and β' are both elements of I, the two sets must be equal because we are iterating over all elements of I, i.e., "$x \in A_\alpha$ for all $\alpha \in I$", and therefore, either way, x still must be contained in both sets so the order does not matter.

(3.4) is true because if y is an element of the intersection of a family of sets, then it must be an element of each and every set in the family of sets. ☐

3.4.10. Suppose that I and J are two sets, and for all $\alpha \in I \cup J$ let A_α be a set. Show that $\left(\bigcup_{\alpha \in I} A_\alpha\right) \cup \left(\bigcup_{\alpha \in J} A_\alpha\right) = \bigcup_{\alpha \in I \cup J} A_\alpha$. If I and J are non-empty, show that $\left(\bigcap_{\alpha \in I} A_\alpha\right) \cap \left(\bigcap_{\alpha \in J} A_\alpha\right) = \bigcap_{\alpha \in I \cup J} A_\alpha$.

Proof.
$\left[\left(\bigcup_{\alpha \in I} A_\alpha\right) \cup \left(\bigcup_{\alpha \in J} A_\alpha\right) = \bigcup_{\alpha \in I \cup J} A_\alpha\right]$:

If $x \in \left(\bigcup_{\alpha \in I} A_\alpha\right) \cup \left(\bigcup_{\alpha \in J} A_\alpha\right)$ then $x \in \left(\bigcup_{\alpha \in I} A_\alpha\right)$ or $x \in \left(\bigcup_{\alpha \in J} A_\alpha\right)$ and therefore $x \in A_\alpha$ for some $\alpha \in I$ or $x \in A_\alpha$ for some $\alpha \in J$. Thus $x \in A_\alpha$ for some $\alpha \in I \cup J$ and hence $x \in \bigcup_{\alpha \in I \cup J} A_\alpha$. Conversely, if $x \in \bigcup_{\alpha \in I \cup J} A_\alpha$ then $x \in A_\alpha$ for some $\alpha \in I \cup J$ and therefore $x \in A_\alpha$ for some $\alpha \in I$ or $x \in A_\alpha$ for some $\alpha \in J$. Thus $x \in \left(\bigcup_{\alpha \in I} A_\alpha\right)$ or $x \in \left(\bigcup_{\alpha \in J} A_\alpha\right)$ and therefore $x \in \left(\bigcup_{\alpha \in I} A_\alpha\right) \cup \left(\bigcup_{\alpha \in J} A_\alpha\right)$. We conclude that $\left(\bigcup_{\alpha \in I} A_\alpha\right) \cup \left(\bigcup_{\alpha \in J} A_\alpha\right) = \bigcup_{\alpha \in I \cup J} A_\alpha$.

$\left[\left(\bigcap_{\alpha\in I} A_\alpha\right) \cap \left(\bigcap_{\alpha\in J} A_\alpha\right) = \bigcap_{\alpha\in I\cup J} A_\alpha\right]$: I and J are non-empty.

If $x \in \left(\bigcap_{\alpha\in I} A_\alpha\right) \cap \left(\bigcap_{\alpha\in J} A_\alpha\right)$ then $x \in \left(\bigcap_{\alpha\in I} A_\alpha\right)$ and $x \in \left(\bigcap_{\alpha\in J} A_\alpha\right)$ and therefore $x \in A_\alpha$ for all $\alpha \in I$ and $x \in A_\alpha$ for all $\alpha \in J$. Thus $x \in A_\alpha$ for all $\alpha \in I \cup J$ and hence $x \in \bigcap_{\alpha\in I\cup J} A_\alpha$. Conversely, if $x \in \bigcap_{\alpha\in I\cup J} A_\alpha$ then $x \in A_\alpha$ for all $\alpha \in I \cup J$ and therefore $x \in A_\alpha$ for all $\alpha \in I$ or $x \in A_\alpha$ for all $\alpha \in J$. Thus $x \in \left(\bigcap_{\alpha\in I} A_\alpha\right)$ and $x \in \left(\bigcap_{\alpha\in J} A_\alpha\right)$ and therefore $x \in \left(\bigcap_{\alpha\in I} A_\alpha\right) \cap \left(\bigcap_{\alpha\in J} A_\alpha\right)$. We conclude that $\left(\bigcap_{\alpha\in I} A_\alpha\right) \cap \left(\bigcap_{\alpha\in J} A_\alpha\right) = \bigcap_{\alpha\in I\cup J} A_\alpha$. ☐

3.4.11. Let X be a set, let I be a non-empty set, and for all $\alpha \in I$ let A_α be a subset of X. Show that

$$X \setminus \bigcup_{\alpha\in I} A_\alpha = \bigcap_{\alpha\in I} (X \setminus A_\alpha)$$

and

$$X \setminus \bigcap_{\alpha\in I} A_\alpha = \bigcup_{\alpha\in I} (X \setminus A_\alpha)$$

This should be compared with De Morgan's laws in Proposition 3.1.28 (although one cannot derive the above identities directly from De Morgan's laws, as I could be infinite).

Proof.
$\left[X \setminus \bigcup_{\alpha\in I} A_\alpha = \bigcap_{\alpha\in I} (X \setminus A_\alpha)\right]$:

If $x \in X\setminus\bigcup_{\alpha\in I} A_\alpha$ then $x \in X$ and $x \notin \bigcup_{\alpha\in I} A_\alpha$ and hence $x \in X$ and $x \notin A_\alpha$ for all $\alpha \in I$. Thus $x \in X\setminus A_\alpha$ for all $\alpha \in I$ and therefore $x \in \bigcap_{\alpha\in I} (X \setminus A_\alpha)$. Conversely, if $x \in \bigcap_{\alpha\in I} (X \setminus A_\alpha)$ then $x \in X\setminus A_\alpha$ for all $\alpha \in I$ and hence $x \in X$ and $x \notin A_\alpha$ for all $\alpha \in I$. Thus $x \in X$ and $x \notin \bigcup_{\alpha\in I} A_\alpha$ and therefore $x \in X \setminus \bigcup_{\alpha\in I} A_\alpha$. We conclude that $X \setminus \bigcup_{\alpha\in I} A_\alpha = \bigcap_{\alpha\in I} (X \setminus A_\alpha)$.

$\left[X \setminus \bigcap_{\alpha\in I} A_\alpha = \bigcup_{\alpha\in I} (X \setminus A_\alpha)\right]$:

If $x \in X \setminus \bigcap_{\alpha\in I} A_\alpha$ then $x \in X$ and $x \notin \bigcap_{\alpha\in I} A_\alpha$ and hence $x \in X$ and $x \notin A_\alpha$ for some $\alpha \in I$. Thus $x \in X \setminus A_\alpha$ for some $\alpha \in I$ and therefore $x \in \bigcup_{\alpha\in I} (X \setminus A_\alpha)$. Conversely, if $x \in \bigcup_{\alpha\in I} (X \setminus A_\alpha)$ then $x \in X \setminus A_\alpha$ for some $\alpha \in I$ and hence $x \in X$ and $x \notin A_\alpha$ for some $\alpha \in I$. Thus $x \in X$ and $x \notin \bigcap_{\alpha\in I} A_\alpha$ and therefore $x \in X \setminus \bigcap_{\alpha\in I} A_\alpha$. We conclude that $X \setminus \bigcap_{\alpha\in I} A_\alpha = \bigcup_{\alpha\in I} (X \setminus A_\alpha)$. ☐

§3.5 Cartesian products

3.5.1. Suppose we define the ordered pair (x, y) for any objects x and y by the formula $(x, y) := \{\{x\}, \{x, y\}\}$ (thus using several applications of Axiom 3.3). Thus for instance $(1, 2)$ is the set $\{\{1\}, \{1, 2\}\}, (2, 1)$ is the set $\{\{2\}, \{2, 1\}\}$, and $(1, 1)$ is the set $\{\{1\}\}$. Show that such a definition indeed obeys the property (3.5), and also whenever X and Y are sets, the Cartesian product $X \times Y$ is also a set. Thus this definition can be validly used as a definition of an ordered pair. For an additional challenge, show that the alternate definition $(x, y) := \{x, \{x, y\}\}$ also verifies (3.5) and is thus also an acceptable definition of ordered pair. (For this latter task one needs the axiom of regularity, and in particular Exercise 3.2.2.)

Proof. Let x, y, x', y' be objects.

If $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$ then by Definition 3.1.4 (set equality) we know that elements of these sets must be equal to each other. Therefore, $\{x\} = \{x'\}$ and $\{x, y\} = \{x', y'\}$ (we would get a contradiction with Definition 3.1.4 if we tried to equate a singleton set with a pair set). Thus, we must have that $x = x'$ and

$y = y'$. Conversely, suppose that $x = x'$ and $y = y'$. Using Axiom 3.3, create singleton sets for x and x' and a pair set for x, y and x', y'. Since sets are objects themselves, use Axiom 3.3 one last time to construct the sets $\{\{x\}, \{x, y\}\}$ and $\{\{x'\}, \{x', y'\}\}$. Then by Definition 3.1.4 must have that $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$. We conclude this definition obeys the property (3.5).

Furthermore, if X and Y are sets then the Cartesian product $X \times Y$ is by definition

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

For this definition of ordered pair we would then have

$$X \times Y = \{\{\{x\}, \{x, y\}\} \mid x \in X, y \in Y\}.$$

and therefore $X \times Y$ is still a set.

Suppose now that the definition for an ordered pair is $(x, y) := \{x, \{x, y\}\}$.

Again, let x, y, x', y' be objects.

If $\{x, \{x, y\}\} = \{x', \{x', y'\}\}$ then by Definition 3.1.4 we know that the elements of these sets must be equal to each other. Therefore, similar to the argument above, using Definition 3.1.4 multiple times we arrive at the conclusion that $x = x'$ and $y = y'$. Conversely, suppose that $x = x'$ and $y = y'$. Using Axiom 3.3, create a pair set for x, y and x', y'. Since sets are objects themselves, use Axiom 3.3 one last time to construct the sets $\{x, \{x, y\}\}$ and $\{x', \{x', y'\}\}$. Then by Definition 3.1.4 must have that $\{x, \{x, y\}\} = \{x', \{x', y'\}\}$. We conclude this definition obeys the property (3.5). □

3.5.2. Suppose we define an ordered n-tuple to be a surjective function $x : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to X$ whose range is some arbitrary set X (so different ordered n-tuples are allowed to have different ranges); we then write x_i for $x(i)$, and also write x as $(x_i)_{1 \leq i \leq n}$. Using this definition, verify that we have $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$ if and only if $x_i = y_i$ for all $1 \leq i \leq n$. Also, show that if $(X_i)_{1 \leq i \leq n}$ are an ordered n-tuple of sets, then the Cartesian product, as defined in Definition 3.5.7, is indeed a set. (Hint: use Exercise 3.4.7 and the axiom of specification.)

Proof. If $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$ then by definition we have that $(x(i))_{1 \leq i \leq n} = (y(i))_{1 \leq i \leq n}$ since we wrote x_i for $x(i)$ and similarly y_i for $y(i)$. Thus, $x(i) = y(i)$ for all $1 \leq i \leq n$ and therefore $x_i = y_i$ for all $1 \leq i \leq n$. Conversely, if $x_i = y_i$ for all $1 \leq i \leq n$ then $x(i) = y(i)$ for all $1 \leq i \leq n$ and thus $(x(i))_{1 \leq i \leq n} = (y(i))_{1 \leq i \leq n}$. Since we wrote x_i for $x(i)$ and similarly y_i for $y(i)$ we have that $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$, by definition. Therefore, $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$ if and only if $x_i = y_i$.

If $(X_i)_{1 \leq i \leq n}$ are an ordered n-tuple of sets, then as is shown in the textbook on p. 59 where if one has some set I, and for every element $\alpha \in I$ we have some set A_α, then we can form the union set $\bigcup_{\alpha \in I} A_\alpha$ by defining

$$\bigcup_{\alpha \in I} A_\alpha := \bigcup \{A_\alpha : \alpha \in I\}.$$

We see that we have this same situation with $I = \{i \in \mathbf{N} : 1 \leq i \leq n\}$ and the sets X_i in $(X_i)_{1 \leq i \leq n}$. Thus, we have $\bigcup \{X_i \mid i \in I\}$. This set is a set of all the elements x_i for all X_i. Now we define the function $f : I \to \bigcup \{X_i \mid i \in I\}$ such that $f(i) = x_i$, for some $x_i \in X_i$. Then, using the axiom of replacement we construct the set $\{f \mid f(i) \in X_i \text{ for all } i \in I\}$, which is equal to the Cartesian product. □

3.5.3. Show that the definitions of equality for ordered pair and ordered n-tuple obey the reflexivity, symmetry, and transitivity axioms.

Proof. TODO – must have missed this one. □

3.5.4. Let A, B, C be sets. Show that $A\times(B\cup C) = (A\times B)\cup(A\times C)$, that $A\times(B\cap C) = (A\times B)\cap(A\times C)$, and that $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$ (One can of course prove similar identities in which the rôles of the left and right factors of the Cartesian product are reversed.)

Proof.
$[A \times (B \cup C) = (A \times B) \cup (A \times C)]$:

If $x \in A \times (B \cup C)$ then $x \in \{(a, b) \mid a \in A, b \in B\}$ or $x \in \{(a, c) \mid a \in A, c \in C\}$ and hence $x \in A \times B$ or $x \in A\times C$. Thus, $x \in (A\times B)\cup(A\times C)$. Conversely, if $x \in (A\times B)\cup(A\times C)$ then $x \in A\times B$ or $x \in A\times C$ and hence $x \in \{(a, b) \mid a \in A, b \in B\}$ or $x \in \{(a, c) \mid a \in A, c \in C\}$. Thus, $x \in A \times (B \cup C)$. Therefore, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

$[A \times (B \cap C) = (A \times B) \cap (A \times C)]$:

If $x \in A \times (B \cap C)$ then $x \in \{(a, b) \mid a \in A, b \in B\}$ and $x \in \{(a, c) \mid a \in A, c \in C\}$ and hence $x \in A \times B$ and $x \in A \times C$. Thus, $x \in (A \times B) \cap (A \times C)$. Conversely, if $x \in (A \times B) \cap (A \times C)$ then $x \in A \times B$ and $x \in A \times C$ and hence $x \in \{(a, b) \mid a \in A, b \in B\}$ and $x \in \{(a, c) \mid a \in A, c \in C\}$. Thus, $x \in A \times (B \cap C)$. Therefore, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

$[A \times (B \setminus C) = (A \times B) \setminus (A \times C)]$:

If $x \in A \times (B \setminus C)$ then $x \in \{(a, b) \mid a \in A, b \in B\}$ and $x \notin \{(a, c) \mid a \in A, c \in C\}$ and hence $x \in A \times B$ and $x \notin A \times C$. Thus, $x \in (A \times B) \setminus (A \times C)$. Conversely, if $x \in (A \times B) \setminus (A \times C)$ then $x \in A \times B$ and $x \notin A \times C$ and hence $x \in \{(a, b) \mid a \in A, b \in B\}$ and $x \notin \{(a, c) \mid a \in A, c \in C\}$. Thus, $x \in A \times (B \setminus C)$. Therefore, $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$. □

3.5.5. Let A, B, C, D be sets. Show that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. Is it true that $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$? Is it true that $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$?

Proof.
$[(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)]$:

If $x \in (A \times B) \cap (C \times D)$ then $x \in A \times B$ and $x \in C \times D$ and hence $x \in \{(a, b) \mid a \in A, b \in B\}$ and $x \in \{(c, d) \mid c \in C, d \in D\}$. Thus, $x \in \{(r, s) \mid r \in A\cap C, s \in B\cap D\}$ so that $x \in (A\cap C)\times(B\cap D)$. Conversely, if $x \in (A \cap C) \times (B \cap D)$ then $x \in \{(r, s) \mid r \in A \cap C, s \in B \cap D\}$ and hence $x \in \{(a, b) \mid a \in A, b \in B\}$ and $x \in \{(c, d) \mid c \in C, d \in D\}$. Thus, $x \in A \times B$ and $x \in C \times D$ so that $x \in (A \times B) \cap (C \times D)$. Therefore, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Is it true that $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$? No.

Is it true that $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$? No. □

3.5.6. Let A, B, C, D be non-empty sets. Show that $A \times B \subseteq C \times D$ if and only if $A \subseteq C$ and $B \subseteq D$, and that $A \times B = C \times D$ if and only if $A = C$ and $B = D$. What happens if the hypotheses that the A, B, C, D are all non-empty are removed?

Proof. If $A \times B \subseteq C \times D$ then for $x \in A \times B$ we have that $x \in C \times D$. Thus, the first element of x is in both A and C and therefore we have that $A \subseteq C$. The same argument holds for the second element of x and therefore $B \subseteq D$. Conversely, if $A \subseteq C$ and $B \subseteq D$ we must have that if $x \in A \times B$ then $x \in C \times D$. Thus, $A \times B \subseteq C \times D$.

If $A \times B = C \times D$ then $\{(a,b) \mid a \in A, b \in B\} = \{(c,d) \mid c \in C, d \in D\}$ and therefore $A = C$ and $B = D$. Conversely, if $A = C$ and $B = D$ then $\{(a,b) \mid a \in A, b \in B\} = \{(c,d) \mid c \in C, d \in D\}$ and therefore $A \times B = C \times D$.

What happens if the hypotheses that the A, B, C, D are all non-empty are removed?

The Cartesian product of any non-empty set with the empty set is the empty set. Note that this is different than the empty Cartesian product mentioned in Example 3.5.11.

For $A \times B \subseteq C \times D$ if and only if $A \subseteq C$ and $B \subseteq D$, this wouldn't make sense if $C = D = \emptyset$ while A and B were non-empty so we can see that there are situations that lead to absurdities. We see similar situations for the second statement as well. □

3.5.7. Let X, Y be sets, and let $\pi_{X\times Y\to X} : X \times Y \to X$ and $\pi_{X\times Y\to Y} : X \times Y \to Y$ be the maps $\pi_{X\times Y\to X}(x,y) := x$ and $\pi_{X\times Y\to Y}(x,y) := y$; these maps are known as the co-ordinate functions on $X \times Y$. Show that for any functions $f : Z \to X$ and $g : Z \to Y$, there exists a unique function $h : Z \to X \times Y$ such that $\pi_{X\times Y\to X} \circ h = f$ and $\pi_{X\times Y\to Y} \circ h = g$. (Compare this to the last part of Exercise 3.3.8, and to Exercise 3.1.7.) This function h is known as the direct sum of f and g and is denoted $h = f \oplus g$.

Proof. TODO – must have missed this one. □

3.5.8. Let $X_1, \ldots, X_n$ be sets. Show that the Cartesian product $\prod_{i=1}^n X_i$ is empty if and only if at least one of the X_i is empty.

Proof. By Definition 3.5.7 we have that $\prod_{i=1}^n X_i := \{(x_i)_{1\le i\le n} \mid x_i \in X_i \text{ for all } 1 \le i \le n\}$.

If at least one of the X_i is empty then we must have that $(x_i)_{1\le i\le n}$ does not exist, for if it did that would imply that $x_i \in X_i$ exists, which is a contradiction. Thus, $\prod_{i=1}^n X_i$ is empty. Conversely, if $\prod_{i=1}^n X_i$ is empty then this means that $(x_i)_{1\le i\le n}$ does not exist which can only happen if at least one of the X_i is empty. Therefore, the Cartesian product $\prod_{i=1}^n X_i$ is empty if and only if at least one of the X_i is empty. □

3.5.9. Suppose that I and J are two sets, and for all $\alpha \in I$ let A_α be a set, and for all $\beta \in J$ let B_β be a set. Show that $\left(\bigcup_{\alpha\in I} A_\alpha\right) \cap \left(\bigcup_{\beta\in J} B_\beta\right) = \bigcup_{(\alpha,\beta)\in I\times J} (A_\alpha \cap B_\beta)$.

Proof. If $x \in \left(\bigcup_{\alpha\in I} A_\alpha\right) \cap \left(\bigcup_{\beta\in J} B_\beta\right)$ then $x \in \left(\bigcup_{\alpha\in I} A_\alpha\right)$ and $x \in \left(\bigcup_{\beta\in J} B_\beta\right)$. Thus, $x \in \bigcup_{(\alpha,\beta)\in I\times J} (A_\alpha \cap B_\beta)$, as we are summing over all combinations of α and β and therefore $I \times J$. Conversely, if $x \in \bigcup_{(\alpha,\beta)\in I\times J} (A_\alpha \cap B_\beta)$, as we are summing over all combinations of α and β and therefore $I \times J$, then $x \in \left(\bigcup_{\alpha\in I} A_\alpha\right)$ and $x \in \left(\bigcup_{\beta\in J} B_\beta\right)$. Thus, $x \in \left(\bigcup_{\alpha\in I} A_\alpha\right) \cap \left(\bigcup_{\beta\in J} B_\beta\right)$. Therefore, $\left(\bigcup_{\alpha\in I} A_\alpha\right) \cap \left(\bigcup_{\beta\in J} B_\beta\right) = \bigcup_{(\alpha,\beta)\in I\times J} (A_\alpha \cap B_\beta)$. □

3.5.10. If $f : X \to Y$ is a function, define the graph of f to be the subset of $X \times Y$ defined by $\{(x, f(x)) : x \in X\}$. Show that two functions $f : X \to Y, \tilde{f} : X \to Y$ are equal if and only if they have the same graph. Conversely, if G is any subset of $X \times Y$ with the property that for each $x \in X$ the set $\{y \in Y : (x,y) \in G\}$ has exactly one element (or in other words, G obeys the vertical line test), show that there is exactly one function $f : X \to Y$ whose graph is equal to G.

Proof. If $f : X \to Y, \tilde{f} : X \to Y$ are equal, then $f(x) = \tilde{f}(x)$ for all $x \in X$ by Definition 3.3.7. Thus, $\{(x, f(x)) \mid x \in X\} = \{(x, \tilde{f}(x)) \mid x \in X\}$ and therefore they have the same graph. Conversely, if f and $\tilde{f}$

have the same graph then $\{(x, f(x)) \mid x \in X\} = \{(x, \tilde{f}(x)) \mid x \in X\}$. Thus, $f(x) = \tilde{f}(x)$ for all $x \in X$ and by Definition 3.3.7, $f : X \to Y, \tilde{f} : X \to Y$ are equal. Therefore, two functions $f : X \to Y, \tilde{f} : X \to Y$ are equal if and only if they have the same graph.

If G is any subset of $X \times Y$ with the property that for each $x \in X$ the set $\{y \in Y \mid (x, y) \in G\}$ has exactly one element (or in other words, G obeys the vertical line test), then let $f : X \to Y$ be the unique function that has the graph $\{(x, y) \mid x \in X, y \in \{y \in Y \mid (x, y) \in G\}\}$ (note that f is unique because we have defined it using its graph). This graph is equal to G. □

3.5.11. Show that Axiom 3.10 can in fact be deduced from Lemma 3.4.9 and the other axioms of set theory, and thus Lemma 3.4.9 can be used as an alternate formulation of the power set axiom. (Hint: for any two sets X and Y, use Lemma 3.4.9 and the axiom of specification to construct the set of all subsets of $X \times Y$ which obey the vertical line test. Then use Exercise 3.5.10 and the axiom of replacement.)

Proof. Let's break up the hint in to pieces to make this more manageable. First, using Lemma 3.4.9 we will construct the set of all subsets of $X \times Y$:

$$X' \times Y' = \{Z \mid Z \subseteq (X \times Y)\}$$

Then, using the axiom of specification let the property $P(Z)$ be true if for each $x \in X$ the set $\{y \in Y \mid (x, y) \in Z\}$ has exactly one element.

That is, we are creating the set

$$X'' \times Y'' = \{Z \in X' \times Y' \mid P(Z) \text{ is true}\}.$$

Thus, the elements of $X'' \times Y''$ are the subsets of $X \times Y$ such that they obey the vertical line test.

Using this fact and the result from Exercise 3.5.10, which showed there was a unique function for each $G \subseteq X \times Y$ (note we are using Z instead of G) that had the property that we used with the axiom of specification above, we can now use the axiom of replacement to create the power set for all the functions from X to Y.

$$Y^X = \{f \mid f \text{ is the function for the graph } Z\}.$$

Therefore, Axiom 3.10 can in fact be deduced from Lemma 3.4.9 and the other axioms of set theory, and thus Lemma 3.4.9 can be used as an alternate formulation of the power set axiom.

3.5.12. This exercise will establish a rigorous version of Proposition 2.1.16. Let $f : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ be a function, and let c be a natural number. Show that there exists a function $a : \mathbf{N} \to \mathbf{N}$ such that

$$a(0) = c$$

and

$$a(n{+}{+}) = f(n, a(n)) \text{ for all } n \in \mathbf{N}$$

and furthermore that this function is unique. (Hint: first show inductively, by a modification of the proof of Lemma 3.5.12, that for every natural number $N \in \mathbf{N}$, there exists a unique function $a_N : \{n \in \mathbf{N} : n \leq N\} \to \mathbf{N}$ such that $a_N(0) = c$ and $a_N(n{+}{+}) = f(n, a(n))$ for all $n \in \mathbf{N}$ such that $n < N$.) For an additional challenge, prove this result without using any properties of the natural numbers other than the Peano axioms directly (in particular, without using the ordering of the natural numbers, and without appealing to Proposition 2.1.16). (Hint: first show inductively, using only the Peano axioms and basic set theory, that for every natural number $N \in \mathbf{N}$, there exists a unique pair A_N, B_N of subsets of $\mathbf{N}$ which obeys the following properties: (a) $A_N \cap B_N = \emptyset$, (b) $A_N \cup B_N = \mathbf{N}$, (c) $0 \in A_N$, (d) $N+ \in B_N$, (e) Whenever $n \in B_N$, we have $n+ \in B_N$. (f) Whenever $n \in A_N$ and $n \neq N$, we have $n + \epsilon \in A_N$. Once one obtains these sets, use A_N as a substitute for $\{n \in \mathbf{N} : n \leq N\}$ in the previous argument.)

We will follow the suggested hint and induct on $N \in \mathbf{N}$.

base case: Let $N = 0$. Since there are no natural numbers less than N, the base case is vacuously true.

induction hypothesis: Suppose that for every natural number $N \in \mathbf{N}$, there exists a unique function $a_N : \{n \in \mathbf{N} : n \leq N\} \to \mathbf{N}$ such that $a_N(0) = c$ and $a_N(n{+}{+}) = f(n, a(n))$ for all $n \in \mathbf{N}$ such that $n < N$.

induction step: Now we will show this is also true for $N{+}{+}$. Since $N{+}{+} = N + 1$, we can use the same formula as the induction hypothesis and we only need to check the case of $n{+}{+}$, as the induction hypothesis covers $n < N$. That is, we check to see iff $a_{N++}((n{+}{+}){+}{+}) = f(n{+}{+}, a(n{+}{+}))$ exists and is unique. We can see that from the induction hypothesis that $a(n{+}{+})$ exists and is unique and thus $f(n{+}{+}, a(n{+}{+}))$ must exist, which makes it unique as it is a function. Therefore $a_{N++}((n{+}{+}){+}{+})$ exists and is unique, closing the induction.

Additional challenge left to reader (may come back to fill this part in). □

3.5.13. The purpose of this exercise is to show that there is essentially only one version of the natural number system in set theory (cf. the discussion in Remark 2.1.12). Suppose we have a set $\mathbf{N}'$ of "alternative natural numbers", an "alternative zero" $0'$, and an "alternative increment operation" which takes any alternative natural number $n' \in \mathbf{N}'$ and returns another alternative natural number $n'{+}{+}' \in \mathbf{N}'$, such that the Peano axioms (Axioms 2.1-2.5) all hold with the natural numbers, zero, and increment replaced by their alternative counterparts. Show that there exists a bijection $f : \mathbf{N} \to \mathbf{N}'$ from the natural numbers to the alternative natural numbers such that $f(0) = 0'$, and such that for any $n \in \mathbf{N}$ and $n' \in \mathbf{N}'$, we have $f(n) = n'$ if and only if $f(n{+}{+}) = n'{+}{+}'$. (Hint: use Exercise 3.5.12.)

Proof. To show this we will basically be doing the same induction as was done in Exercise 3.5.12. However, instead of having a two parameter function f used in the recursive definition of the function A_N, here we have a one parameter function f that is contained in the recursive statement $f(n) = n'$ if and only if $f(n{+}{+}) = n'{+}{+}'$.

base case: Let $n = 0$ such that $n{+}{+} = 1$. Then $f(0) = 0'$ and $f(0{+}{+}) = 0'{+}{+} \implies f(1) = 1'$.

induction hypothesis: Suppose that there exists a bijection $f : \mathbf{N} \to \mathbf{N}'$ from the natural numbers to the alternative natural numbers such that $f(0) = 0'$, and such that for any $n \in \mathbf{N}$ and $n' \in \mathbf{N}'$, we have $f(n) = n'$ if and only if $f(n{+}{+}) = n'{+}{+}'$.

induction step: Now we will show that this is true for $n{+}{+}$. From the induction hypothesis we know that we already have a bijection $f : \mathbf{N} \to \mathbf{N}'$ from the natural numbers to the alternative natural numbers such that $f(0) = 0'$, and such that for any $n \in \mathbf{N}$ and $n' \in \mathbf{N}'$, we have $f(n) = n'$ if and only if $f(n{+}{+}) = n'{+}{+}'$. We need to show that $f(n{+}{+}) = n{+}{+}'$ if and only if $f((n{+}{+}){+}{+}) = (n{+}{+})'{+}{+}'$. From the induction hypothesis we have that $f(n{+}{+}) = n'{+}{+}'$ and therefore we must have the consequent of the implication, namely $f((n{+}{+}){+}{+}) = (n{+}{+})'{+}{+}'$, closing the induction. □

§3.6 Cardinality of sets

3.6.1. Prove Proposition 3.6.4.

Proof. Let X, Y, Z be sets.

(reflexive): Let $f : X \to X$ such that $f(x) = x$. This is a bijective map and therefore X has equal cardinality with X.

(symmetric): If X has equal cardinality with Y there must exist a bijective function $f : X \to Y$ and therefore it is invertible and its inverse $f^{-1} : Y \to X$ is also a bijection. Therefore, Y has equal cardinality with X.

(transitive): If X has equal cardinality with Y and Y has equal cardinality with Z, then there exist bijective maps, say $f : X \to Y$ and $g : Y \to Z$ and therefore their composition $g \circ f$ is also a bijective map (composition of bijective functions is bijective). □

3.6.2. Show that a set X has cardinality 0 if and only if X is the empty set.

Proof. Suppose a set X has cardinality 0. Then by Definition 3.6.5 X has 0 elements and therefore X is the empty set. Conversely, if X is the empty set then it has zero elements and by Definition 3.6.5 X must have cardinality 0. Therefore, a set X has cardinality 0 if and only if X is the empty set. □

3.6.3. Let n be a natural number, and let $f : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to \mathbf{N}$ be a function. Show that there exists a natural number M such that $f(i) \leq M$ for all $1 \leq i \leq n$. (Hint: induct on n. You may also want to peek at Lemma 5.1.14.) Thus finite subsets of the natural numbers are bounded.

Proof.
base case: Let $n = 0$. Then $f : \{i \in \mathbf{N} : 1 \leq i \leq 0\} \to \mathbf{N}$ is a function from the empty set to $\mathbf{N}$ and therefore $f(i) \leq M$ for a natural number M is vacuously true.

induction hypothesis: Let $f : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to \mathbf{N}$ and suppose that there exists a natural number M such that $f(i) \leq M$ for all $1 \leq i \leq n$.

induction step: Now we will show that this is also true for $n+1$. Let $f : \{i \in \mathbf{N} : 1 \leq i \leq n+1\} \to \mathbf{N}$ and from induction hypothesis we know there exists a natural number M such that $f(i) \leq M$ for all $1 \leq i \leq n$. Let $M' = M + f(n+1)$, which we know must be a natural number since both M and $f(n+1)$ are natural numbers. Then, we have $f(i) \leq M'$ for all $1 \leq i \leq n+1$, closing the induction. Thus finite subsets of the natural numbers are bounded. □

3.6.4. Prove Proposition 3.6.14.

Proof.

(a) Let X be a finite set, and let x be an object which is not an element of X. Then $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$.

By Definition 3.6.10 the finite set X has cardinality n for some natural number n which we denote $\#(X)$. Likewise, by Definition 3.6.5 the singleton set $\{x\}$, since it has 1 element, has cardinality 1. Therefore, $\#(X \cup \{x\}) = \#(X) + 1$.

(b) Let X and Y be finite sets. Then $X \cup Y$ is finite and $\#(X \cup Y) \leq \#(X) + \#(Y)$. If in addition X and Y are disjoint (i.e., $X \cap Y = \emptyset$), then $\#(X \cup Y) = \#(X) + \#(Y)$.

If the finite sets X and Y have any elements in common then we know that the number of elements in $X \cup Y$ is less than the sum of the individual number of elements in X and Y. Therefore $\#(X \cup Y) \leq \#(X) + \#(Y)$.

If X and Y are disjoint (i.e., $X \cap Y = \emptyset$), then obviously $\#(X \cup Y) = \#(X) + \#(Y)$.

(c) Let X be a finite set, and let Y be a subset of X. Then Y is finite, and $\#(Y) \leq \#(X)$. If in addition $Y \neq X$ (i.e., Y is a proper subset of X), then we have $\#(Y) < \#(X)$.

The number of elements in a subset is equal or less than the number of elements in the set of which it is contained. Therefore, if X is a finite set and $Y \subseteq X$ then we have that $\#(Y) \leq \#(X)$. In the event that $Y \neq X$ (i.e., Y is a proper subset of X), then Y must have less elements than X has and therefore $\#(Y) < \#(X)$.

(d) If X is a finite set, and $f : X \to Y$ is a function, then $f(X)$ is a finite set with $\#(f(X)) \leq \#(X)$. If in addition f is one-to-one, then $\#(f(X)) = \#(X)$.

For a function $f : X \to Y$ the image, $f(X)$ may be either equal or less than the size of the domain X. The reason for this is that the function must be well defined but we may have the case that multiple elements of the domain map to the same elements of the range Y. Therefore, in general we have that $\#(f(X)) \leq \#(X)$. However, in the event that f is one-to-one we know that for each element in X it must be mapped to a unique element in $f(X)$ and therefore they both have the same amount of elements so that $\#(f(X)) = \#(X)$.

(e) Let X and Y be finite sets. Then Cartesian product $X \times Y$ is finite and $\#(X \times Y) = \#(X) \times \#(Y)$.

By definition 3.5.4 of the Cartesian product we know that for each element in X we have $\#(Y)$ tuples and therefore $\#(X) \times \#(Y)$ tuples in total. Therefore $\#(X \times Y) = \#(X) \times \#(Y)$.

(f) Let X and Y be finite sets. Then the set Y^X (defined in Axiom 3.10) is finite and $\#\left(Y^X\right) = \#(Y)^{\#(X)}$.

From Axiom 3.10 we also know that $f \in Y^X \iff$ (f is a function with domain X and range Y). For each element of X there is $\#(Y)$ different elements of Y that this element can be mapped to. That is, for a given element of X there are $\#(Y)$ different functions that would map this element to the $\#(Y)$ different elements of Y. Since there are $\#(X)$ elements in X we see that there are $\#(Y)^{\#(X)}$ functions from X to Y and therefore $\#(Y^X) = \#(Y)^{\#(X)}$.

□

3.6.5. Let A and B be sets. Show that $A \times B$ and $B \times A$ have equal cardinality by constructing an explicit bijection between the two sets. Then use Proposition 3.6.14 to conclude an alternate proof of Lemma 2.3.2.

Proof. We have the sets $A \times B = \{(a, b) \mid a \in A, b \in B\}$ and $B \times A = \{(b, a) \mid b \in B, a \in A\}$. Let f be the the map from $A \times B$ to $B \times A$ such that $f((a, b)) = (b, a)$. That is, the tuple order is reversed. The inverse function then would revere the order again back to the original. Thus, f is a bijection and therefore we must have that $\#(A \times B) = \#(B \times A)$.

Lemma 2.3.2 claims that multiplication is commutative for the natural numbers. Using $\#(A \times B) = \#(B \times A)$ and Proposition 3.6.14(e), we see that $\#(A) \times \#(B) = \#(B) \times \#(A)$, thereby giving us an alternate proof of Lemma 2.3.2. □

3.6.6. Let A, B, C be sets. Show that the sets $\left(A^B\right)^C$ and $A^{B \times C}$ have equal cardinality by constructing an explicit bijection between the two sets. Conclude that $\left(a^b\right)^c = a^{bc}$ for any natural numbers a, b, c. Use a similar argument to also conclude $a^b \times a^c = a^{b+c}$.

Proof. Note that

$$(A^B)^C = \{f = g' \circ g \mid g : C \to \{g' \mid g' : B \to A\}\}$$
$$A^{B \times C} = \{f \mid f : B \times C \to A\}$$

Each of these sets contain functions that only depend on a specific choice of a, b, c. Therefore, let $f : (A^B)^C \to A^{B \times C}$ be the bijection that for a specific choice of a, b, c maps the function in $(A^B)^C$ to the function in $A^{B \times C}$ that uses the same choice of a, b, c. This is a bijection because we can simply perform the inverse map to get back to the original function in the domain.

Therefore, $\#\left(\left(A^B\right)^C\right) = \#\left(A^{B \times C}\right)$. Using this result and Proposition 3.6.14(e,f) we can conclude that $\left(a^b\right)^c = a^{bc}$ for any natural numbers a, b, c. The same goes for $a^b \times a^c = a^{b+c}$ (we are using 3.6.14(a) and the result of 3.6.14(f) multiple times until we arrive at the forms we have here). □

3.6.7. Let A and B be sets. Let us say that A has *lesser or equal* cardinality to B if there exists an injection $f : A \to B$ from A to B. Show that if A and B are finite sets, then A has lesser or equal cardinality to B if and only if $\#(A) \leq \#(B)$

Proof. Let A and B be finite sets.

Suppose that A has lesser or equal cardinality to B. Then there exists an injection $f : A \to B$ from A to B. Since f is injective we know that the number of elements in the range, B, must contain at least or more of the number of elements in the domain, A. Thus, $\#(A) \leq \#(B)$. Conversely, if $\#(A) \leq \#(B)$ then the number of elements in B must contain at least or more of the number of elements in A. Let us construct a map from A to B where for every element in A we assign this single element to a unique element in B (this can be constructed in theory as any explicit map would suffice). Therefore, there exists an injection $f : A \to B$. Thus, A has lesser or equal cardinality to B.

Therefore, if A and B are finite sets, then A has lesser or equal cardinality to B if and only if $\#(A) \leq \#(B)$. □

3.6.8. Let A and B be sets such that there exists an injection $f : A \to B$ from A to B (i.e., A has lesser or equal cardinality to B). Show that there then exists a surjection $g : B \to A$ from B to A. (The converse to this statement requires the axiom of choice; see Exercise 8.4.3.)

Proof. Since A has lesser or equal cardinality to B there exists an injection $f : A \to B$ from A to B and therefore, from Exercise 3.6.7 above we know that $\#(A) \leq \#(B)$. Hence, the number of elements in B is equal to or larger than the number of elements in A. Let us construct a map from B to A where every element in A is mapped to, from B, at least once (this can be constructed in theory as any explicit map would suffice). Therefore, there exists a surjection $g : B \to A$ from B to A. □

3.6.9. Let A and B be finite sets. Show that $A \cup B$ and $A \cap B$ are also finite sets, and that $\#(A) + \#(B) = \#(A \cup B) + \#(A \cap B)$.

Proof. If A and B are finite sets. Then, since $A \cap B \subseteq A$ or $A \cap B \subseteq B$, we must have that $A \cap B$ is finite. Let us denote the cardinality of $A \cap B$ with the natural number s, i.e., $\#(A \cap B) = s$.

Additionally, since $\#(A) = n$ and $\#(B) = m$ for some natural numbers n, m, we see that the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ will have $\#(A \cup B) = m + n - s$ elements and therefore $A \cup B$ is finite. We haven't proved subtraction yet (this will come in the next chapter) so instead of $m + n - s$ let us represent the number of elements in $A \cup B$ as the natural number j such that $m + n = j + s$.

Therefore, we have $\#(A \cup B) + \#(A \cap B) = j + s = m + n = \#(A) + \#(B)$ as desired. □

3.6.10. Let $A_1, \ldots, A_n$ be finite sets such that $\#\left(\bigcup_{i \in \{1,\ldots,n\}} A_i\right) > n$. Show that there exists $i \in \{1, \ldots, n\}$ such that $\#\left(A_i\right) \geq 2$. (This is known as the pigeonhole principle.)

Proof. For $\#\left(\bigcup_{i \in \{1,\ldots,n\}} A_i\right) > n$ suppose there does not exist $i \in \{1, \ldots, n\}$ such that $\#\left(A_i\right) \geq 2$. Thus, $\#(A_i) \leq 1$ for all $i \in \{1, \ldots, n\}$. Since the cardinality of each A_i is either 0 or 1 we have that $\#\left(\bigcup_{i \in \{1,\ldots,n\}} A_i\right) \leq n$, which is a contradiction. Therefore, there exists $i \in \{1, \ldots, n\}$ such that $\#\left(A_i\right) \geq 2$. □