Analysis I
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Chapter 4 - Integers and rationals

Exercises:

§4.1 The integers

4.1.1. Verify that the definition of equality on the integers is both reflexive and symmetric.

Two integers are considered to be equal, $a - b = c - d$, if and only if $a + d = c + b$.

(reflexive): If $a - b = a - b$, then $a + b = a + b$.

(symmetric): If $a - b = c - d$ and $c - d = a - b$, then $a + d = c + b$ and $c + b = a + d$.

Therefore equality on the integers is both reflexive and symmetric.

4.1.2. Show that the definition of negation on the integers is well-defined in the sense that if $(a-b) = (a' - b')$, then $-(a - b) = -(a' - b')$ (so equal integers have equal negations).

Proof. If $(a - b) = (a' - b')$, then $-(a - b) = -(a' - b')$ and by definition we have that $(b - a) = (b' - a')$ which implies $b + a' = b' + a$.

Therefore, negation is well-defined. □

4.1.3. Show that $(-1) \times a = -a$ for every integer a.

Proof. $(-1) = 0 - 1$ and $a = a - 0$. Therefore, $(-1) \times a$ becomes $(0 - 1) \times (a - 0) = (0 \cdot a + 1 \cdot 0) - (0 \cdot 0 + 1 \cdot a) = -a$. □

4.1.4. Prove the remaining identities in Proposition 4.1.6. (Hint: one can save some work by using some identities to prove others. For instance, once you know that $xy = yx$, you get for free that $x1 = 1x$, and once you also prove $x(y + z) = xy + xz$, you automatically get $(y + z)x = yx + zx$ for free.)

Proof. Let use denote $x = (a - b)$, $y = (c - d)$, and $z = (e - f)$.

1. $x + y = y + x$:

$$\begin{aligned} x + y &= y + x \\ (a - b) + (c - d) &= (c - d) + (a - b) \\ (a + c) - (b + d) &= (c + a) - (d + b) \end{aligned}$$

2. $(x + y) + z = x + (y + z)$

$$\begin{aligned} (x + y) + z &= x + (y + z) \\ ((a - b) + (c - d)) + (e - f) &= (a - b) + ((c - d) + (e - f)) \\ ((a + c) - (b + d)) + (e - f) &= (a - b) + ((c + e) - (d + f)) \\ ((a + c) + e) - ((b + d) + f) &= (a + (c + e)) - (b + (d + f)) \end{aligned}$$

3. $x+0=0+x=x$

$$\begin{aligned} x+0=0+x&=x\\ (a-b)+0=0+(a-b)&=(a-b)\\ (a-b)+(0-0)=(0-0)+(a-b)&=(a-b)\\ (a+0)-(b+0)=(0+a)-(0+b)&=(a-b)\\ (a-b)=(a-b)&=(a-b) \end{aligned}$$

4. $x+(-x)=(-x)+x=0$

$$\begin{aligned} x+(-x)=(-x)+x&=0\\ (a-b)+(-(a-b))=(-(a-b))+(a-b)&=0-0\\ (a-b)+(b-a)=(b-a)+(a-b)&=0-0\\ (a+b)-(b+a)=(b+a)-(a+b)&=0-0 \end{aligned}$$

5. $xy=yx$

$$\begin{gathered} xy=yx\\ (a-b)\times(c-d)=(c-d)\times(a-b)\\ (ac+bd)-(ad+bc)=(ca+db)-(da+cb) \end{gathered}$$

6. $(xy)z=x(yz)$

   Completed in the textbook (p. 79).

7. $x1=1x=x$

   Let $1-0=y$ and apply the property $xy=yx$.

8. $x(y+z)=xy+xz$

$$\begin{aligned} x(y+z)&=xy+xz\\ (a-b)\times((c-d)+(e-f))&=(a-b)\times(c-d)+(a-b)\times(e-f)\\ (a-b)\times((c+e)-(d+f))&=(a-b)\times(c-d)+(a-b)\times(e-f)\\ (a(c+e)+b(d+f))-(a(d+f)+b(c+e))&=(ac+bd)-(ad+bc)+(ae+bf)-(af+be)\\ (ac+ae+bd+bf)-(ad+af+bc+be)&=(ac+bd)+(ae+bf)-(ad+bc)-(af+be)\\ (ac+ae+bd+bf)-(ad+af+bc+be)&=(ac+bd+ae+bf)-(ad+bc+af+be) \end{aligned}$$

9. $(y+z)x=yx+zx$

   Since $xy=yx$ and using the last property $x(y+z)=xy+xz$, we must have that $(y+z)x=yx+zx$.

□

4.1.5. Prove Proposition 4.1.8. (Hint: while this proposition is not quite the same as Lemma 2.3.3, it is certainly legitimate to use Lemma 2.3.3 in the course of proving Proposition 4.1.8.)

Proof. Let a and b be integers such that $ab = 0$.

From Lemma 4.1.5 we know that a and b are either equal to: zero, a positive natural number, or the negation of a positive natural number. From Lemma 2.3.3 we know that Proposition 4.1.8 holds for a, b when they are non-negative integers (i.e., when they are natural numbers). Thus, we just need to show that Proposition 4.1.8 also holds for integers that are the negation of a positive natural number and the combinations that this entails. The cases that need to be checked are when either a or b are the negation of a positive natural number or both.

First, as a reference, we will look at the formulation of when a and b are both equal to positive natural numbers (i.e., when Proposition 4.1.8 holds from Lemma 2.3.3). In the case that a, b are equal to positive natural numbers let us denote them as $a = (m - n)$ and $b = (r - s)$ for natural numbers m, n, r, s . Then,

(reference case - where we know that Proposition 4.1.8 is true):

$$\begin{aligned} ab &= 0 \\ (m-n) \times (r-s) &= 0 \\ (mr+ns) - (ms+nr) &= 0 - 0 = 0 && \text{[step 3]} \\ (mr+ns) + 0 &= 0 + (ms+nr) && \text{[step 4]} \end{aligned}$$

(case 1 - both a and b are the negation of positive natural numbers):

This means we will now denote a, b as $a = -(m - n) = (n - m)$ and $b = -(r - s) = (s - r)$. Then,

$$\begin{aligned} ab &= 0 \\ (n-m) \times (s-r) &= 0 \\ (ns+mr) - (nr+ms) &= 0 \end{aligned}$$

which has the same form as step 3 in the reference case above. Thus, Proposition 4.1.8 also holds when a, b are both equal to the negation of positive natural numbers.

(case 2 - a is equal to the negation of a positive natural number but b is equal to a positive natural number):

Thus, $a = -(m - n) = (n - m)$ and $b = (r - s)$. Then,

$$\begin{aligned} ab &= 0 \\ (n-m) \times (r-s) &= 0 \\ (nr+ms) - (ns+mr) &= 0 - 0 \\ (nr+ms) + 0 &= 0 + (ns+mr) \\ 0 + (nr+ms) &= (ns+mr) = 0 \end{aligned}$$

which has the same form as the step 4 in the reference case above. A similar argument holds when b is equal to the negation of a positive natural number.

Therefore, if a and b are integers such that $ab = 0$. Then either $a = 0$ or $b = 0$ (or both). $\square$

4.1.6. Prove Corollary 4.1.9. (Hint: there are two ways to do this. One is to use Proposition 4.1.8 to conclude that $a - b$ must be zero. Another way is to combine Corollary 2.3.7 with Lemma 4.1.5.)

Proof. If a, b, c are integers such that $ac = bc$ and c is non-zero, then

$$\begin{aligned} ac &= bc \\ ac + 0 &= bc + 0 \end{aligned}$$

$$ac - bc = 0 - 0$$
$$c(a-b) = 0$$

Since c is non-zero, by Proposition 4.1.8 we must have that $a - b = 0$ and therefore

$$a - b = 0 - 0 = 0$$
$$a + 0 = 0 + b$$
$$a = b$$

□

4.1.7. Prove Lemma 4.1.11. (Hint: use the first part of this lemma to prove all the others.)

Proof. Let a, b, c be integers.

(a) *$a > b$ if and only if $a - b$ is a positive natural number.*

If $a > b$ then $a = b + n$ for some natural number n. Thus, $a - b = n$ and therefore $a - b$ is a positive natural number. Conversely, if $a - b$ is a positive natural number let it be n. Thus, $a - b = n$ so that $a = b + n$ and therefore $a > b$.

(b) *(Addition preserves order) If $a > b$, then $a + c > b + c$.*

If $a > b$ then $a = b + c$ for some positive natural number c. Adding c to both sides gives $a + c = b + 2c > b + c$. Therefore, $a + c > b + c$.

(c) *(Positive multiplication preserves order) If $a > b$ and c is positive, then $ac > bc$.*

If $a > b$ and c is positive, then $a > b \implies a = b + n$ for some positive natural number n. Multiplying both sides by c we have that $ac = (b + n)c = bc + nc > bc$ and therefore $ac > bc$.

(d) *(Negation reverses order) If $a > b$, then $-a < -b$.*

If $a > b$ then $a = b + n$ for positive natural number n. Thus, $a - b = n$ and taking the negation of both sides (multiplying by -1) we have that $-(a - b) = -n \implies (b - a) = -n$. Hence $-a = -n - b$ and therefore $-a = -(n + b) < -b$.

(e) *(Order is transitive) If $a > b$ and $b > c$, then $a > c$.*

If $a > b$ and $b > c$ then $a = b + n_1$ and $b = c + n_2$, with n_1, n_2 being positive natural numbers. Thus $a = b + n + 1 = (c + n_2) + n_1 = c + (n_2 + n_1)$ and therefore $a > c$.

(f) *(Order trichotomy) Exactly one of the statements $a > b, a < b$, or $a = b$ is true.*

$a > b$ and $a < b$ cannot simultaneously be true, nor can $a > b$ and $a = b$, nor $a < b$ and $a = b$. Thus, one of them must be true.

□

4.1.8. Show that the principle of induction (Axiom 2.5) does not apply directly to the integers. More precisely, give an example of a property $P(n)$ pertaining to an integer n such that $P(0)$ is true, and that $P(n)$ implies $P(n++)$ for all integers n, but that $P(n)$ is not true for all integers n. Thus induction is not as useful a tool for dealing with the integers as it is with the natural numbers. (The situation becomes even worse with the rational and real numbers, which we shall define shortly.)

Proof. Let $P(n)$ be the statement that $P(n) = 0 + 1 + \cdots + n = \dfrac{n(n+1)}{2}$. This is obviously true for the natural numbers but is not true for all the integers because it doesn't make sense for the negative integers. Thus induction is not as useful a tool for dealing with the integers as it is with the natural numbers. (The situation becomes even worse with the rational and real numbers, which we shall define shortly.) $\square$

§4.2 The rationals

4.2.1. Show that the definition of equality for the rational numbers is reflexive, symmetric, and transitive. (Hint: for transitivity, use Corollary 4.1.9.)

Proof.

(reflexive): If $a//b = a//b$, then $ab = ab$.

(symmetric): If $a//b = c//d$ then $ad = bc$ therefore $bc = ad$ so that $c//d = a//d$.

(transitive): If $a//b = c//d$ and $c//d = e//f$, then $ad = bc$ and $cf = de$. Multiplying both equations we have $adcf = bcde$. Using the cancellation property we have that $af = be$ and therefore $a//b = e//f$. $\square$

4.2.2. Prove the remaining components of Lemma 4.2.3.

Proof. We will prove the remaining claims of product and negation operations on rational numbers. Let $a//b = a'//b'$.

(product): We must show that $(a//b) \times (c//d) = (a'//b') \times (c//d)$ and $(c//d) \times (a//b) = (c//d) \times (a'//b')$.

If $(a//b) \times (c//d) = (a'//b') \times (c//d)$ then $(ac)//(bd) = (a'c)//(b'd)$ and since $a//b = a'//b'$, the claim follows. Similarly if one replaces $c//d$ by $c'//d'$.

(negation): $-(a//b) = (-a)//b$ and $-(a'//b') = (-a')//b'$ by definition. Since $a//b = a'//b'$ we see that $(-a)//b = (-a')//b'$ and thus $(-a)b' = (-a')b$, the claim follows. $\square$

4.2.3. Prove the remaining components of Proposition 4.2.4. (Hint: as with Proposition 4.1.6, you can save some work by using some identities to prove others.)

Proof. Let use denote $x = (a//b)$, $y = (c//d)$, and $z = (e//f)$.

1. $x + y = y + x$:

$$x + y = y + x$$
$$(a//b) + (c//d) = (c//d) + (a//b)$$
$$(ad + bc)//bd = (cb + da)//db$$

2. $(x + y) + z = x + (y + z)$

   Completed in the textbook (p. 84).

3. $x + 0 = 0 + x = x$

$$x + 0 = 0 + x = x$$

$$
\begin{aligned}
(a//b) + (0//1) = (0//1) + (a//b) &= (a//b) \\
(a1 + b0)//b1 = (0b + 1a)//1b &= (a//b) \\
(a)//b = (a)//b &= (a//b)
\end{aligned}
$$

4. $x + (-x) = (-x) + x = 0$

$$
\begin{aligned}
x + (-x) = (-x) + x &= 0 \\
(a//b) + (-(a//b)) = (-(a//b)) + (a//b) &= 0 \\
(a//b) + (-a)//b = (-a)//b + (a//b) &= 0 \\
(ab + (-a)b)//bb = ((-a)b + ab)//bb &= 0
\end{aligned}
$$

5. $xy = yx$

$$
\begin{aligned}
xy &= yx \\
(a//b) \times (c//d) &= (c//d) \times (a//b) \\
(ac)//(bd) &= (ca)//(db) \\
(ac)(db) &= (bd)(ca)
\end{aligned}
$$

6. $(xy)z = x(yz)$

$$
\begin{aligned}
(xy)z &= x(yz) \\
((a//b) \times (c//d)) \times (e//f) &= (a//b) \times ((c//d) \times (e//f)) \\
((ac)//(bd)) \times (e//f) &= (a//b) \times ((ce)//(df)) \\
(ace)//(bdf) &= (ace)//(bdf)
\end{aligned}
$$

7. $x1 = 1x = x$

   Let $y = 1$ and apply the property $xy = yx$.

8. $x(y + z) = xy + xz$

$$
\begin{aligned}
x(y + z) &= xy + xz \\
(a//b) \times ((c//d) + (e//f)) &= (a//b) \times (c//d) + (a//b) \times (e//f) \\
(a//b) \times (cf + de)//(df) &= (ac)//(bd) + (ae)//(bf) \\
(a(cf + de))//(bdf) &= ((ac)(bf) + (bd)(ae))//(bbdf) \\
(acf + ade)//(bdf) &= (acf + dae)//(bdf)
\end{aligned}
$$

9. $(y + z)x = yx + zx$

   Since $xy = yx$ and using the last property $x(y + z) = xy + xz$, we must have that $(y + z)x = yx + zx$.

$\square$

4.2.4. Prove Lemma 4.2.7. (Note that, as in Proposition 2.2.13, you have to prove two different things: firstly, that at least one of (a), (b), (c) is true; and secondly, that at most one of (a), (b), (c) is true.)

Proof. As the hint mentions, we first need to prove that *at least* one of (a), (b), (c) is true.

Let x be a rational number.

(a) x is equal to 0.

(b) x is a positive rational number.

(c) x is a negative rational number.

First we prove that *at least* one of the above statements is true.

By definition, $x = a/b$ for some integers a and b. From Lemma 4.1.5 we know that an integer is either equal to zero, a positive natural number, or the negation of a positive natural number. By definition we know that $a/0$ and $0/0$ are undefined so for non-zero rational numbers we are left with the cases that $(-a)/b, a/(-b), (-a)/(-b), a/b$. However, since $(-a)/b = a/(-b)$ by definition and $(-a)/(-b) = a/b$, we are left with three cases: $0/b, (-a)/b, a/b$, which by definition are a rational number equal to zero, a positive rational number, and a negative rational number. Therefore, we must have that at least one of the above statements is true.

Secondly, we prove that *at most* one of the above statements is true.

We obviously can't have that x is equal to 0 and a positive rational number simultaneously as that would imply $0/1 = a/b$ for some positive integers a and b and this leads to a contradiction with $0b = 1a$. We have a similar situation with x being equal to 0 and a negative rational number simultaneously. Furthermore we can't have that x is equal to a positive rational number and a negative rational number simultaneously as this would imply $a/b = (-c)/d$ for some positive integers a, b, c, d and this leads to a contradiction with $ad = (-c)b$. Therefore, we must have that at most one of the above statements is true.

This closes the proof of Lemma 4.2.7. □

4.2.5. Prove Proposition 4.2.9.

Proof. Let x, y, z be rational numbers.

(a) (Order trichotomy) Exactly one of the three statements $x = y, x < y$, or $x > y$ is true.

If $x > y$ then by Definition 4.2.8 we have that $x-y = a/b$ for positive integers a and b. Thus, $x = a/b+y$ so we can't have $x = y$ or $x < y$. The same argument holds when $x < y$. If $x = y$ then for $x = a/b$ and $y = c/d$, where a, b, c, d are integers we have that $a/b = c/d$ and thus $ad = bc$. Hence, we can't have $ad < bc$ and $ad > bc$ so we can't have that $x < y$ or $y > x$. Therefore, exactly one of the three statements $x = y, x < y$, or $x > y$ is true.

(b) (Order is anti-symmetric) One has $x < y$ if and only if $y > x$.

If $x < y$ then $x - y$ is a negative rational number, say $(-a)/b$, for positive integers a and b. Thus,

$$\begin{aligned} x - y &= (-a)/b \\ b(x - y) &= -a \\ b(x - y) + a &= 0 \\ bx - by + a &= 0 \\ a &= by - bx \\ a &= b(y - x) \end{aligned}$$

$$a/b = y - x$$

and therefore $y > x$.

Conversely, if $y > x$ then $y - x$ is a positive rational number, say a/b, for positive integers a and b. Thus,

$$\begin{aligned} y - x &= a/b \\ b(y - x) &= a \\ b(y - x) - a &= 0 \\ by - bx - a &= 0 \\ -a &= bx - by \\ -a &= b(x - y) \\ -a/b &= x - y \end{aligned}$$

and therefore $x < y$.

(c) (Order is transitive) If $x < y$ and $y < z$, then $x < z$.

If $x < y$ and $y < z$ then $x - y = -a/b$ and $y - z = -c/d$ for positive integers a, b, c, d. Thus, $y = z - c/d$ and plugging this into $x - y = -a/b$ we have that $x - (z - c/d) = -a/b$ and hence $x - z = -c/d - a/d$, which is a rational number and therefore $x < z$.

(d) (Addition preserves order) If $x < y$, then $x + z < y + z$.

If $x < y$ then $x - y$ is a negative rational number, say $-a/b$ for positive integers a and b. If we add z to both sides we have that $x - y + z = -a/b + z$ and hence $(x + z) - (y + z) = -a/b$ so that $x + z < y + z$.

(e) (Positive multiplication preserves order) If $x < y$ and z is positive, then $xz < yz$.

If $x < y$ and $z > 0$, then $x - y = -a/b$ for positive integers a and b. Multiplying z on both sides we have that $(x - y)z = -az/b$ and hence $xz - yz = -az/b$ and therefore $xz < yz$.

□

4.2.6. Show that if x, y, z are rational numbers such that $x < y$ and z is negative, then $xz > yz$.

Proof. If x, y, z are rational numbers such that $x < y$ and $z < 0$ then $x - y = -a/b$ for positive integers a and b. Multiplying z on both sides we have that $(x - y)z = -az/b$. Since z is a negative rational number let us denote it as $-e/f$ for positive integers e and f. Then $(x - y)z = -az/b$ becomes $(x - y)z = (-a(-e/f))/b$ so that we have $xz - yz = ((-a)(-e))/(bf)$ and therefore $xz > yz$. □

§4.3 Absolute value and exponentiation

4.3.1. Prove Proposition 4.3.3. (Hint: while all of these claims can be proven by dividing into cases, such as when x is positive, negative, or zero, several parts of the proposition can be proven without such a tedious division into cases. For instance one can use earlier parts of the proposition to prove later ones.)

Proof. Let x, y, z be rational numbers.

(a) *(Non-degeneracy of absolute value) We have* $|x| \geq 0$. *Also,* $|x| = 0$ *if and only if* x *is* 0.

If x is zero then by definition $|x| = 0$. If x is positive then $|x| = x \geq 0$. If x is negative then $|x| = -x = -(-a)/b$ for positive integers a and b. Thus, $|x| = -(-a)/b = a/b \geq 0$. Therefore we have $|x| \geq 0$.

If $|x| = 0$ then by Definition 4.3.1 x is 0. Conversely, if x is zero, then $|x| = 0$.

(b) *(Triangle inequality for absolute value) We have* $|x + y| \leq |x| + |y|$.

If $x + y$ is zero then by definition we have that $0 \leq |x| + |y|$ as we must have that $0 \leq |x|$ and $0 \leq |y|$.

If $x < 0$ and $y < 0$, or $x > 0$ and $y > 0$ then we have that $|x + y| = x + y$ and $|x| + |y| = x + y$.

If $x < 0$ and $y > 0$, or $x > 0$ and $y < 0$ then we have that the sum $x + y$ will be less than the value of $|x| + |y|$ since some quantity is being subtracted in the sum $x + y$ due to x and y not being both negative or both positive. Therefore, we must have that $|x + y| \leq |x| + |y|$.

(c) *We have the inequalities* $-y \leq x \leq y$ *if and only if* $y \geq |x|$. *In particular, we have* $-|x| \leq x \leq |x|$.

If $-y \leq x \leq y$ then we have that both $-y \leq x$ and $x \leq y$. For $-y \leq x$, if we multiply both sides by -1 then we have that $y \geq -x$. Thus, we have $y \geq -x$ and $y \geq x$ (from $x \leq y$) and therefore we must have that $y \geq |x|$.

In particular, letting $y = |x|$ in $-y \leq x \leq y$ we then have $-|x| \leq x \leq |x|$, which is obviously true since $|x| \geq |x|$.

(d) *(multiplication of absolute value) We have* $|xy| = |x||y|$. *In particular,* $|-x| = |x|$.

If either of x or y are equal to zero then we have both sides of $|xy| = |x||y|$ being zero. If $x > 0$ and $y < 0$ then $xy < 0$ and therefore $|xy| = -(xy) > 0$ and $|x||y| = x(-y) = -(xy)$. If $x < 0$ and $y > 0$ then $xy < 0$ and therefore $|xy| = -(xy) > 0$ and $|x||y| = (-x)y = -(xy)$. If both $x, y < 0$ then $|xy| = xy$ and $|x||y| = (-x)(-y) = xy$ while if both $x, y > 0$ then $|xy| = xy$ and $|x||y| = xy$.

Therefore, we have $|xy| = |x||y|$.

In particular, letting $y = -1$ in $|xy| = |x||y|$ we have that

$$\begin{aligned} |x(-1)| &= |x||-1| \\ |(-1)x| &= |x|1 \\ |-x| &= |x| \end{aligned}$$

(e) *(Non-degeneracy of distance) We have* $d(x, y) \geq 0$. *Also,* $d(x, y) = 0$ *if and only if* $x = y$.

The distance between x and y is the quantity $|x - y|$ also denoted $d(x, y)$. Let $a = x - y$, then $|x - y| = |a|$ and from *Non-degeneracy of absolute value* we must have that $|a| \geq 0$. Therefore we have $d(x, y) \geq 0$.

If $d(x, y) = 0$ then $|x - y| = 0$ and thus $|a| = 0$ and again, by part *(Non-degeneracy of absolute value)* we must have that $a = 0$ and thus $x - y = 0$ and therefore $x = y$.

(f) *(Symmetry of distance)* $d(x, y) = d(y, x)$.

$|x - y| = |y - x|$ since by *multiplication of absolute value*, in particular we have that $|-x| = |x|$. Thus, $d(x, y) = d(y, x)$.

(g) *(Triangle inequality for distance)* $d(x, z) \leq d(x, y) + d(y, z)$.

Let $a = x - y$ and $b = y - z$ so that $a + b = (x - y) + (y - z) = x - z$. From *Triangle inequality for absolute value* we know that $|a + b| \leq |a| + |b|$, therefore

$$\begin{aligned} |a + b| &\leq |a| + |b| \\ |x - z| &\leq |x - y| + |y - z| \\ d(x, z) &\leq d(x, y) + d(y, z) \end{aligned}$$

□

4.3.2. Prove the remaining claims in Proposition 4.3.7.

Proof. Let x, y, z, w be rational numbers.

(a) *If $x = y$, then x is ε-close to y for every $\varepsilon > 0$. Conversely, if x is ε-close to y for every $\varepsilon > 0$, then we have $x = y$.*

If $x = y$, then $d(x, y) = 0$ and therefore $d(x, y) = 0 \leq \varepsilon$. That is x is ε-close to y for every $\varepsilon > 0$. Conversely, if x is ε-close to y for every $\varepsilon > 0$, then $d(x, y) \leq \varepsilon$ and since ε is arbitrary we can make it as small as we want, showing that we must have $x = y$.

(b) *Let $\varepsilon > 0$. If x is ε-close to y, then y is ε-close to x.*

If x is ε-close to y, then $d(x, y) \leq \varepsilon$ and with the *symmetry of distance* we know that $d(x, y) = d(y, x)$. Thus, we must have that $d(y, x) \leq \varepsilon$ and therefore y is ε-close to x.

(c) *Let $\varepsilon, \delta > 0$. If x is ε-close to y, and y is δ-close to z, then x and z are $(\varepsilon + \delta)$-close.*

If x is ε-close to y, and y is δ-close to z, then $d(x, y) \leq \varepsilon$ and $d(y, z) \leq \delta$. Thus, $d(x, y) + d(y, z) \leq \varepsilon + \delta$. From *triangle inequality for distance* we know that $d(x, z) \leq d(x, y) + d(y, z)$ and hence $d(x, z) \leq d(x, y) + d(y, z) \leq (\varepsilon + \delta)$. Therefore, x and z are $(\varepsilon + \delta)$-close.

(d) *Let $\varepsilon, \delta > 0$. If x and y are ε-close, and z and w are δ-close, then $x + z$ and $y + w$ are $(\varepsilon + \delta)$-close, and $x - z$ and $y - w$ are also $(\varepsilon + \delta)$-close.*

If x and y are ε-close, and z and w are δ-close, then $d(x, y) = d(y, x) \leq \varepsilon$ and $d(z, w) = d(w, z) \leq \delta$. Thus, we have that $|x - y| = |y - x|$ and $|z - w| = |w - z|$. From *triangle inequality for absolute value* we then have that $|(x - y) + (z - w)| \leq |x - y| + |z - w| \leq (\varepsilon + \delta)$ and hence $|(x + z) - (y + w)| \leq |x - y| + |z - w| \leq (\varepsilon + \delta)$. Therefore, $x + z$ and $y + w$ are $(\varepsilon + \delta)$-close. Additionally, we see that $|(x - y) + (w - z)| \leq |x - y| + |w - z| \leq (\varepsilon + \delta)$ and hence $|(x - z) - (y - w)| \leq |x - y| + |w - z| \leq (\varepsilon + \delta)$. Therefore, $x - z$ and $y - w$ are also $(\varepsilon + \delta)$-close.

(e) *Let $\varepsilon > 0$. If x and y are ε-close, they are also ε'-close for every $\varepsilon' > \varepsilon$.*

If x and y are ε-close, then $d(x, y) = d(y, x) \leq \varepsilon$ and for every $\varepsilon' > \varepsilon$ we must then have that $d(x, y) = d(y, x) \leq \varepsilon < \varepsilon'$. Therefore, x and y are also ε'-close for every $\varepsilon' > \varepsilon$.

(f) *Let $\varepsilon > 0$. If y and z are both ε-close to x, and w is between y and z (i.e., $y \leq w \leq z$ or $z \leq w \leq y$), then w is also ε-close to x.*

If $y \leq w \leq z$ then $|y - x| \leq |w - x| \leq |z - x| \leq \varepsilon$ and hence $|w - x| \leq \varepsilon$. If $z \leq w \leq y$ then $|z - x| \leq |w - x| \leq |y - x| \leq \varepsilon$ and hence $|w - x| \leq \varepsilon$. Therefore w is also ε-close to x.

(g) *Let $\varepsilon > 0$. If x and y are ε-close, and z is non-zero, then xz and yz are $\varepsilon|z|$-close.*

If x and y are ε-close, and z is non-zero, then $|x-y| = |y-x| \leq \varepsilon$ and hence $|x-y||z| = |y-x||z| \leq \varepsilon|z|$. Thus, $|xz - yz| = |yz - xz| \leq \varepsilon|z|$ and therefore xz and yz are $\varepsilon|z|$-close.

(h) *Let $\varepsilon, \delta > 0$. If x and y are ε-close, and z and w are δ-close, then xz and yw are $(\varepsilon|z|+\delta|x|+\varepsilon\delta)$-close.*

Proved in the textbook (p. 88).

□

4.3.3. Prove Proposition 4.3.10. (Hint: use induction.)

Proof. Let x, y be rational numbers, and let n, m be natural numbers.

(a) *We have $x^n x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, and $(xy)^n = x^n y^n$.*

($x^n x^m = x^{n+m}$): We will use induction on n. For $n = 0$ we have $x^0 x^m = 1x^m = x^m = x^{0+m}$. Suppose the claim holds for n. We will now show it also holds for $n+1$.

$$\begin{aligned} x^{n+1}x^m &= x^n x x^m && \text{[def.]} \\ &= x^n x^m x && \text{[comm.]} \\ &= x^{n+m} x && \text{[induction hyp.]} \\ &= x^{n+m+1} && \text{[def.]} \end{aligned}$$

This closes the induction and therefore, we have $x^n x^m = x^{n+m}$.

($(xy)^n = x^n y^n$): We will use induction on n. For $n = 0$ we have

$$\begin{aligned} (xy)^0 &= x^0 y^0 \\ 1 &= 1 \end{aligned}$$

Suppose the claim holds for n. We will now show it also holds for $n+1$.

$$\begin{aligned} x^{n+1}y^{n+1} &= x^n x y^n y && \text{[def.]} && = x^n y^n x y && \text{[comm.]} \\ &= (xy)^n xy && \text{[induction hyp.]} \\ &= (xy)^{n+1} && \text{[def.]} \end{aligned}$$

This closes the induction and therefore, we have $(xy)^n = x^n y^n$.

($(x^n)^m = x^{nm}$): We will use induction on n. For $n = 0$ we have

$$\begin{aligned} (x^0)^m &= x^{0m} \\ (1)^m &= x^0 \\ 1 &= 1 \end{aligned}$$

Suppose the claim holds for n. We will now show it also holds for $n+1$.

$$\begin{aligned} (x^{n+1})^m &= (x^n x)^m && \text{[def.]} \\ &= x^{nm} x^m && [(xy)^n = x^n y^n] \\ &= x^{(n+1)m} \end{aligned}$$

This closes the induction and therefore, we have $(x^n)^m = x^{nm}$.

(b) *Suppose $n > 0$. Then we have $x^n = 0$ if and only if $x = 0$.*

Let $n > 0$. We will use induction on n. For $n = 1$, if $x = 0$ then $x^n = 0^1 = 0$. Conversely, if $x^1 = 0$ then $x = 0$. Suppose the claim holds for n. We will now show it also holds for $n + 1$. If $x = 0$ then

$$\begin{aligned} x^{n+1} &= x^n x && \text{[def.]} \\ &= 0x && \text{[induction hyp.]} \\ &= 0 \end{aligned}$$

Conversely, if $x^{n+1} = 0$ then

$$\begin{aligned} x^{n+1} &= 0 \\ x^n x &= 0 && \text{[def.]} \end{aligned}$$

and by the induction hypothesis, we know that $x^n = 0$ and $x = 0$. This closes the induction.

Therefore, for $n > 0$ we have $x^n = 0$ if and only if $x = 0$.

(c) *If $x \geq y \geq 0$, then $x^n \geq y^n \geq 0$. If $x > y \geq 0$ and $n > 0$, then $x^n > y^n \geq 0$.*

We will use induction on n for both claims.

For $n = 0$, if $x \geq y \geq 0$, then $x^0 \geq y^0 = 1 \geq 1 \geq 0$. Suppose the claim holds for n. We will now show it also holds for $n + 1$. If $x \geq y \geq 0$, then $x^{n+1} \geq y^{n+1} = x^n x \geq y^n y$. From the induction hypothesis we have that if $x \geq y \geq 0$, then $x^n \geq y^n \geq 0$. Since $x \geq y$ and $x^n \geq y^n$ we must have that $x^n x \geq y^n y$ and also that since $y \geq 0$ and $y^n \geq 0$ that $y^n y \geq 0$. Therefore, we must have that $x^{n+1} \geq y^{n+1} \geq 0$. This closes the induction.

For $n = 1$, if $x > y \geq 0$, then $x^1 > y^1 = x > y \geq 0$. Suppose the claim holds for n. We will now show it also holds for $n + 1$. If $x > y \geq 0$, then $x^{n+1} > y^{n+1} = x^n x > y^n y$. From the induction hypothesis we know that if $x > y \geq 0$, then $x^n > y^n \geq 0$. Since $x > y$ and $x^n > y^n$ we must have that $x^n x > y^n y$ and also since $y \geq 0$ and $y^n \geq 0$ we must have that $y^n y \geq 0$. Therefore, we must have that $x^{n+1} > y^{n+1} \geq 0$. This closes the induction.

(d) *We have $|x^n| = |x|^n$.*

We will use induction on n. For $n = 0$ we have,

$$\begin{aligned} \left|x^0\right| &= |x|^0 \\ 1 &= 1 \end{aligned}$$

Suppose that the claim holds for n. Now we will show it also holds for $n + 1$.

$$\begin{aligned} \left|x^{n+1}\right| &= |x|^{n+1} \\ |x^n x| &= |x|^n |x| \\ |x^n|\,|x| &= |x|^n |x| \end{aligned}$$

and by the induction hypothesis we know that $|x^n| = |x|^n$ so we must have that $\left|x^{n+1}\right| = |x|^{n+1}$. This closes the induction.

□

4.3.4. Prove Proposition 4.3.12. (Hint: induction is not suitable here. Instead, use Proposition 4.3.10.)

Proof. Let x, y be non-zero rational numbers, and let n, m be integers.

(a) *We have* $x^n x^m = x^{n+m}, (x^n)^m = x^{nm}$, *and* $(xy)^n = x^n y^n$.

($x^n x^m = x^{n+m}$): Let $n = -a$ and $m = -b$ for natural numbers a, b. Then $x^n x^m = x^{n+m}$ becomes

$$\begin{aligned} x^{-a}x^{-b} &= x^{-a-b} \\ \frac{1}{x^a x^b} &= \frac{1}{x^{a+b}} \\ x^{a+b} &= x^a x^b \end{aligned}$$

and from Proposition 4.3.10(a), since a, b are natural numbers, we know this is true, therefore showing us that $x^n x^m = x^{n+m}$ holds for negative integers.

($(x^n)^m = x^{nm}$): Let $n = -a$ and $m = -b$ for natural numbers a, b. Then $(x^n)^m = x^{nm}$ becomes

$$\begin{aligned} \left(x^{-a}\right)^{-b} &= x^{(-a)(-b)} \\ \frac{1}{\left(x^{-a}\right)^b} &= x^{ab} \\ \frac{1}{\left(\frac{1}{x^a}\right)^b} &= x^{ab} \\ \frac{1/1}{\frac{1}{\left(x^a\right)^b}} &= x^{ab} \\ \left(x^a\right)^b &= x^{ab} \end{aligned}$$

and from Proposition 4.3.10(a), since a, b are natural numbers, we know this is true, therefore showing us that $(x^n)^m = x^{nm}$ holds for negative integers.

($(xy)^n = x^n y^n$): Let $n = -a$ for natural number a. Then $(xy)^n = x^n y^n$ becomes

$$\begin{aligned} (xy)^{-a} &= x^{-a}y^{-a} \\ \frac{1}{(xy)^a} &= \frac{1}{x^a y^a} \\ x^a y^a &= (xy)^a \end{aligned}$$

and from Proposition 4.3.10(a), since a is a natural number, we know this is true, therefore showing us that $(xy)^n = x^n y^n$ holds for negative integers.

(b) *If* $x \geq y > 0$, *then* $x^n \geq y^n > 0$ *if* n *is positive, and* $0 < x^n \leq y^n$ *if* n *is negative.*

If $x \geq y > 0$, then $x^n \geq y^n > 0$ if n is positive was proven for Proposition 4.3.10(c). For positive n we have $x^n \geq y^n > 0$ and thus $x^n \geq y^n$. Hence, $0 < \frac{1}{x^n} \leq \frac{1}{y^n}$ and therefore, when n is negative we have that $0 < x^n \leq y^n$.

(c) *If* $x, y > 0, n \neq 0$, *and* $x^n = y^n$, *then* $x = y$.

If $x, y > 0, n \neq 0$, and $x^n = y^n$, then

$$x^n = y^n$$

$$
\begin{aligned}
(x^n)^{1/n} &= (y^n)^{1/n} \\
x^{n/n} &= y^{n/n} \\
x &= y
\end{aligned}
$$

(d) *We have* $|x^n| = |x|^n$.

We proved the non-negative portion of this in the proof of Proposition 4.3.10(d). Now let us show it also holds for n a negative integer. Let $n = -b$ for some natural number b. Then, $|x^n| = |x|^n$ becomes

$$
\begin{aligned}
\left|x^{-b}\right| &= |x|^{-b} \\
\left|\frac{1}{x^b}\right| &= \frac{1}{|x|^b} \\
\frac{1}{|x^b|} &= \frac{1}{|x|^b} \\
|x|^b &= \left|x^b\right|
\end{aligned}
$$

and since b is a natural number we know this is true from Proposition 4.3.10(d), showing that when n is a negative integer that $|x^n| = |x|^n$ still holds.

$\square$

4.3.5. Prove that $2^N \geq N$ for all positive integers N. (Hint: use induction.)

Proof. We will use induction on N.

For $N = 1$ we have $2^1 = 2 \geq 1$. Suppose that the claim holds for N. We will now show it also holds for $N + 1$. $2^{N+1} = 2^N 2$ and by the induction hypothesis we know that $2^N \geq N$ and therefore we must have that $2^N 2 \geq 2N \geq N + 1$ since $N > 0$. Therefore, $2^N \geq N$ for all positive integers N. $\square$

§4.4 Gaps in the rational numbers

4.4.1. Prove Proposition 4.4.1. (Hint: use Proposition 2.3.9.)

Proof. Let x be a rational number. We need to show that there exists an integer n such that $n \leq x < n+1$.

Let $x \geq 0$. Then $x = a/b$ where a and b are positive integers. From Lemma 4.1.5 we know that these positive integers are equal to natural numbers. From Proposition 2.3.9 there exist natural numbers m, r such that $a = mb + r$ and $0 \leq r < b$. Since $0 \leq r < b$ we have that $mb \leq mb + r < mb + b = (m + 1)b$. Since $b > 0$ we must have that $1/b > 0$ and hence

$$
\begin{aligned}
&\frac{1}{b}(mb) \leq \frac{1}{b}(mb + r) < \frac{1}{b}((m + 1)b) \\
&m \leq m + r/b < m + 1 \\
&m \leq a/b < m + 1 \\
&m \leq x < m + 1
\end{aligned}
$$

Thus, since m is a natural number we have that there exists an integer n such that $n \leq x < n+1$, by letting $n = m$.

Now let $x < 0$. Then $x = (-a)/b$ where a and b are positive integers. From above, we know that $m \leq -x < m+1$ and therefore $-(m+1) < x \leq -m$. If $x \neq -m$ then let $n = -(m+1)$ so that $n+1 = -m = x$ and therefore we have that $n \leq x < n+1$ (note how using this n has switched the inequalities). If $x = -m$ then let $n = -m$ so that $n = x < n+1$ is true.

Next we show uniqueness of n from $n \leq x < n+1$. For the sake of contradiction suppose that n is not unique and that there exists an integer m such that $n \neq m$ and $m \leq x < m+1$. Since $n \neq m$ we must have that $n < m$ or $n > m$. Without loss of generality suppose that $n < m$. Then $n < m \leq x < n+1$ which shows that $n < m < n+1$. However, n, m and $n+1$ are integers and hence this is a contradiction since there are no integers in between n and $n+1$. Therefore, n for $n \leq x < n+1$ is unique.

This shows that given any rational number x, that in particular we have $N = n+1$ such that $N > x$. $\square$

4.4.2. A definition: a sequence $a_0, a_1, a_2, \ldots$ of numbers (natural numbers, integers, rationals, or reals) is said to be in infinite descent if we have $a_n > a_{n+1}$ for all natural numbers n (i.e., $a_0 > a_1 > a_2 > \ldots$).

(a) Prove the principle of infinite descent: that it is not possible to have a sequence of natural numbers which is in infinite descent. (Hint: assume for sake of contradiction that you can find a sequence of natural numbers which is in infinite descent. Since all the a_n are natural numbers, you know that $a_n \geq 0$ for all n. Now use induction to show in fact that $a_n \geq k$ for all $k \in \mathbf{N}$ and all $n \in \mathbf{N}$, and obtain a contradiction.)

Proof. Let us assume for the sake of contradiction that you can find a sequence, say a_n, of natural numbers which is in infinite descent. Then, since a_n are natural numbers we know that $a_n \geq 0$ for all n. Now we will show using induction that $a_n \geq k$ for all $k \in \mathbf{N}$ and all $n \in \mathbf{N}$.

We will us induction on k. For $k = 0$ we have that $a_n \geq 0$ for all $n \in \mathbf{N}$ which is true by definition. Suppose that the claim holds for k, i.e., that $a_n \geq k$ for all $n \in \mathbf{N}$. We will now show it also holds for $k+1$ and for all $n \in \mathbf{N}$. From the induction hypothesis we know that $a_n \geq k$ for all $n \in \mathbf{N}$ and therefore we must have that $a_{n+1} \geq k$. However, since $a_n > a_{n+1}$ we must have that $a_n - 1 \geq a_{n+1}$. Thus, with $a_{n+1} \geq k$ and $a_n - 1 \geq a_{n+1}$ we have that $a_n - 1 \geq k$ and therefore $a_n \geq k+1$. This closes the induction, showing us that there are no sequences of infinite descent, a contradiction. $\square$

(b) Does the principle of infinite descent work if the sequence $a_1, a_2, a_3, \ldots$ is allowed to take integer values instead of natural number values? What about if it is allowed to take positive rational values instead of natural numbers? Explain.

The principle of infinite descent can work for integers, see the proof of Proposition 4.4.4, which also allowed x to be negative (see the part where $x^2 = (-x)^2$). Since rational numbers are constructed from a ratio of integers it will depend on how the argument is constructed. However, in general, with the rational numbers you can continuously divide a number as many times as you want and you still have another rational number and therefore the principle of infinite descent will not work with the rational numbers as a whole.

4.4.3. Fill in the gaps marked (why?) in the proof of Proposition 4.4.4

Every natural number is either even or odd, but not both:

Because even numbers are a multiple of 2 while odd numbers are all the natural numbers that are not a multiple of 2.

If p is odd, then p^2 is also odd:

$p = 2k+1$ then $p^2 = (2k+1)(2k+1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ which itself an odd number.

Since $p^2 = 2q^2$, *we have* $q < p$:

Since $p^2 = 2q^2$, we have $q < p$ since p^2 is a double (multiplied by 2) of q^2. That is, $q = \sqrt{\frac{p^2}{2}} = \frac{p}{\sqrt{2}}$.