Analysis I
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Chapter 5 - The real numbers

Exercises:

§5.1 Cauchy sequences

5.1.1. Prove Lemma 5.1.15. (Hint: use the fact that a_n is eventually 1-steady, and thus can be split into a finite sequence and a 1-steady sequence. Then use Lemma 5.1.14 for the finite part. Note there is nothing special about the number 1 used here; any other positive number would have sufficed.)

Proof. We need to show that every Cauchy sequence $(a_n)_{n=1}^{\infty}$ is bounded.

By Definition 5.1.8 a sequence is a Cauchy sequence iff for every $\varepsilon > 0$, there exists an $N \geq 0$ such that $d(a_j, a_k) \leq \varepsilon$ for all $j, k \geq N$. Definition 5.1.12 says that an infinite sequence $(a_n)_{n=1}^{\infty}$ is bounded by M iff $|a_i| \leq M$ for all $i \geq 1$.

As the hint suggests, let us use the fact that a_n is eventually 1-steady. For the finite portion of a_n (which is disjoint from the 1-steady part of the sequence) we know from Lemma 5.1.14 that it is bounded by some $M \geq 0$. If the entire sequence is 1-steady to begin with then let $M = |a_1| + 1$ (see Example 5.1.10). Then, for the rest of a_n which is 1-steady, we know that the difference between successive elements of the sequence will not be more than 1. Thus, the 1-steady portion of the sequence is also bounded by M.

Therefore, every Cauchy sequence $(a_n)_{n=1}^{\infty}$ is bounded. □

§5.2 Equivalent Cauchy sequences

5.2.1. Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence if and only if $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Proof. If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, then for each rational $\varepsilon > 0$ the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close. That is, for every rational $\varepsilon > 0$, there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.

Now, if $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence then it is eventually ε-steady. That is, there exists an $N \geq 0$ such that $d(a_j, a_k) \leq \varepsilon$ for all $j, k \geq N$. Thus, we know that for ever rational $\varepsilon > 0$ there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ and $d(a_j, a_k) = |a_j - a_k| \leq \varepsilon$ for all $n, j, k \geq N$. Hence, we can re-label n so that we have $|a_j - b_j|$ or $|a_k - b_k|$. Flipping one of the absolute values, summing, and using the triangle inequality for absolute value we have

$$|(b_j - a_j) + (a_k - b_k)| \leq |b_j - a_j| + |a_k - b_k| \leq \varepsilon + \varepsilon \quad (1)$$

$$|(b_j - b_k) + (a_k - a_j)| \leq \quad (2)$$

$$|(b_j - b_k)| \leq |(b_j - b_k) + (a_k - a_j)| \leq \quad (3)$$

We see that $|(b_j - b_k)| \leq \varepsilon + \varepsilon = \varepsilon'$ for all $j, k \geq N$ showing that $(b_n)_{n=1}^{\infty}$ is also a Cauchy sequence. The converse argument reasons the same way.

Therefore, if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence if and only if $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence. □

5.2.2. Let $\varepsilon > 0$. Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, then $(a_n)_{n=1}^{\infty}$ is bounded if and only if $(b_n)_{n=1}^{\infty}$ is bounded.

Proof. If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, then there exists an $N \geq 0$ such that the sequences $(a_n)_{n=N}^{\infty}$ and $(b_n)_{n=N}^{\infty}$ are ε-close. That is, there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.

Now, if $(a_n)_{n=1}^{\infty}$ is bounded then it is bounded by M for some rational $M \geq 0$. That is, $|a_i| \leq M$ for all $i \geq 1$. Since $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$ and $|a_n - b_n| = |b_n - a_n|$ because $d(a_n, b_n) = d(b_n, a_n)$ we see that

$$|b_n| = |(b_n - a_n) + a_n| \leq |b_n - a_n| + |a_n| \leq \varepsilon + M$$

for all $n \geq N$. Now, $|b_n|$ for $n < N$ is bounded, by say M_2, since it is a finite sequence (Lemma 5.1.14). Therefore, $|b_i| \leq \varepsilon + M + M_2$ for all $i \geq 1$, showing us that $(b_n)_{n=1}^{\infty}$ is bounded. The converse argument reasons the same way.

Therefore, if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, then $(a_n)_{n=1}^{\infty}$ is bounded if and only if $(b_n)_{n=1}^{\infty}$ is bounded. □

§5.3 The construction of the real numbers

5.3.1. Prove Proposition 5.3.3. (Hint: you may find Proposition 4.3.7 to be useful.)

Proof. Let $x = \mathrm{LIM}_{n\to\infty} a_n, y = \mathrm{LIM}_{n\to\infty} b_n, z = \mathrm{LIM}_{n\to\infty} c_n$.

Since any sequence is 0-close to itself the sequence $(a_n)_{n=1}^{\infty}$ is 0-close to itself and therefore $(a_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ are equivalent sequences. Thus, if $x = \mathrm{LIM}_{n\to\infty} a_n$ we must have that $x = x$ as the two Cauchy sequences $(a_n)_{n=1}^{\infty}$ are equivalent.

If $x = y$ then $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$ so that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences. Thus, we must have that $\mathrm{LIM}_{n\to\infty} b_n = \mathrm{LIM}_{n\to\infty} a_n$ and therefore $y = x$.

If $x = y$ and $y = z$ then $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$ and $\mathrm{LIM}_{n\to\infty} b_n = \mathrm{LIM}_{n\to\infty} c_n$ so that $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences. Now, since $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$, and $(c_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences we must have that $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} c_n$ and therefore $x = z$. □

5.3.2. Prove Proposition 5.3.10. (Hint: again, Proposition 4.3.7 may be useful.)

Proof. Let $x = \mathrm{LIM}_{n\to\infty} a_n, y = \mathrm{LIM}_{n\to\infty} b_n$ and $x' = \mathrm{LIM}_{n\to\infty} a'_n$. By Definition 5.3.9 we have that $xy := \mathrm{LIM}_{n\to\infty} a_n b_n$. Now we will show that this is a real number. By Definition 5.3.1 a real number is defined to be an object of the form $\mathrm{LIM}_{n\to\infty} a_n$, where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Therefore, we need to show that $(a_n b_n)_{n=1}^{\infty}$ is a Cauchy sequence. Once again, a Cauchy sequence is a sequence that is eventually ε-steady and a sequence that is eventually ε-steady is a sequence where for any rational $\varepsilon > 0$ there exists $N \geq 1$ such that $|a_j b_j - a_k b_k| \leq \varepsilon$ for all $n \geq N$ ($N \geq 1$ since we are indexing the sequence starting at 1). Now, from the definition of real numbers we know that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are Cauchy sequences, that is, that they are eventually ε-steady and δ-steady, respectively. Therefore we must have that for any $\varepsilon, \delta > 0$ there exists $N_1, N_2 \geq 1$ such that $|a_j - a_k| \leq \varepsilon$ and $|b_s - b_t| \leq \delta$ for all $j, k \geq N_1$

and $s, t \geq N_2$. Let $N = \max(N_1, N_2)$ so that $j, k, s, t \geq N$. Since we are now indexing over the same N we can just use j and k in place of s and t for example. From Proposition 4.3.7(h) we know that since a_j, a_k are ε-close and b_j, b_k are δ-close, then $a_j b_j$ and $a_k b_k$ are both $(\varepsilon|b_j| + \delta|a_j| + \varepsilon\delta)$-close. Since ε and δ are arbitrarily small, we can take them to be values such that $a_j b_j$ and $a_k b_k$ are both eventually ε'-close. This shows that $(a_n b_n)_{n=1}^{\infty}$ is a Cauchy sequence and therefore xy is a real number.

Furthermore, if $x = x'$ and we follow the same construction as above we will have two sequences, xy and $x'y$, that must be equivalent Cauchy sequences as they are both eventually ε'-close for some $\varepsilon' > 0$. □

5.3.3. Let a, b be rational numbers. Show that $a = b$ if and only if $\mathrm{LIM}_{n\to\infty} a = \mathrm{LIM}_{n\to\infty} b$ (i.e., the Cauchy sequences $a, a, a, a, \ldots$ and $b, b, b, b \ldots$ equivalent if and only if $a = b$). This allows us to embed the rational numbers inside the real numbers in a well-defined manner.

Proof. If $a = b$ then we have that $a, a, a, a \cdots = b, b, b, b \ldots$ Both sequences are Cauchy and therefore we have that $\mathrm{LIM}_{n\to\infty} a = \mathrm{LIM}_{n\to\infty} b$. Conversely, if $\mathrm{LIM}_{n\to\infty} a = \mathrm{LIM}_{n\to\infty} b$ then $a, a, a, a \ldots$ and $b, b, b, b \ldots$ are both Cauchy sequences. Thus, they are eventually ε-close and therefore by Proposition 4.3.7(a) we must have that $a = b$. □

5.3.4. Let $(a_n)_{n=0}^{\infty}$ be a sequence of rational numbers which is bounded. Let $(b_n)_{n=0}^{\infty}$ be another sequence of rational numbers which is equivalent to $(a_n)_{n=0}^{\infty}$. Show that $(b_n)_{n=0}^{\infty}$ is also bounded. (Hint: use Exercise 5.2.2.)

Proof. Let $(a_n)_{n=0}^{\infty}$ be bounded by M.

Since $(a_n)_{n=0}^{\infty}$ is equivalent to $(b_n)_{n=0}^{\infty}$ we have that for any $\varepsilon > 0$ there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$. Similar to Exercise 5.2.2 we see that since $|a_n - b_n| = |b_n - a_n|$ we can show that $|b_n| = |(b_n - a_n) + a_n| \leq |b_n - a_n| + |a_n| \leq \varepsilon + M$ and therefore $(b_n)_{n=0}^{\infty}$ is also bounded. □

5.3.5. Show that $\mathrm{LIM}_{n\to\infty} 1/n = 0$.

Proof. For $\mathrm{LIM}_{n\to\infty} 1/n = 0$ we need to show that for any $\varepsilon > 0$ that there exists an $N \geq 0$ such that $|1/n - 0| = |1/n| \leq \varepsilon$ for all $n \geq N$. We can let $\varepsilon = 1/n$ as $|1/n| \leq 1/n$ for all $n \geq N$. Thus, we have that $\mathrm{LIM}_{n\to\infty} 1/n = 0$. □

§5.4 Ordering the reals

5.4.1. Prove Proposition 5.4.4. (Hint: if x is not zero, and x is the formal limit of some sequence $(a_n)_{n=1}^{\infty}$, then this sequence cannot be eventually ε-close to the zero sequence $(0)_{n=1}^{\infty}$ for every single $\varepsilon > 0$. Use this to show that the sequence $(a_n)_{n=1}^{\infty}$ is eventually either positively bounded away from zero or negatively bounded away from zero.)

Proof. Proposition 5.4.4 claims *For every real number x, exactly one of the following three statements is true: (a) x is zero; (b) x is positive; (c) x is negative. A real number x is negative if and only if $-x$ is positive. If x and y are positive, then so are $x + y$ and xy.*

We need to show that *at least* one of $(a) - (c)$ is true and also that *at most* one of $(a) - (c)$ is true.

If $x = 0$ then we have (a). If $x \neq 0$ then $x = \mathrm{LIM}_{n\to\infty} a_n$ for some sequence of rational numbers $(a_n)_{n=1}^{\infty}$. Now, since $x \neq 0$ we know that $(a_n)_{n=1}^{\infty}$ must be bounded away from zero by Lemma 5.3.14. By Definition

5.4.1 we must then have that $(a_n)_{n=1}^{\infty}$ is either positively or negatively bounded away from zero. Thus, by Definition 5.4.3 we must have that x is either positive or negative. Thus, exactly one of $(a) - (c)$ is true.

If a real number x is negative then we must have that the sequence of its formal limit is negatively bounded away from zero. Thus, there exists a negative rational $-c < 0$ such that $a_n \leq -c$ for all $n \geq 1$. By Proposition 5.3.11 we know that $-x$ would give us $\text{LIM}_{n\to\infty} -a_n$ and therefore we would have that $c > 0$ such that $-a_n \geq c$ for all $n \geq 1$. Thus, $-x$ is positively bounded away from zero and therefore it must be positive. The converse argument is performed in the same manner in the opposite direction. Therefore, x is negative if and only if $-x$ is positive.

If x and y are positive, then x and y are both positively bounded away from zero. Then, using Proposition 5.3.11 we see that the sum or multiplication of these sequences must also be positively bounded away from zero and therefore $x + y$ and xy are also positive. □

5.4.2. Prove the remaining claims in Proposition 5.4.7.

Proof. Let x, y, z be real numbers.

(a) (*Order trichotomy*) *Exactly one of the three statements $x = y, x < y$, or $x > y$ is true.*

If $x > y$ then by Definition 5.4.6 we have that $x - y$ is a positive real number. Thus, $x - y \neq 0$ and from Proposition 5.4.4 if $x - y$ is positive then $-(x - y) = y - x$ is negative. If $x < y$ then use the previous argument but switch the positions of x and y. If $x = y$ then $x - y = 0$ so we can't have either of $x > y$ or $x < y$. Therefore, exactly one of the three statements $x = y, x < y$, or $x > y$ is true.

(b) (*Order is anti-symmetric*) *One has $x < y$ if and only if $y > x$.*

If $x < y$ then $x - y$ is a negative real number and by Proposition 5.4.4 we must have that $-(x-y) = y-x$ is a positive number so that $y > x$. The converse argument is the same but in the other direction. Therefore, one has $x < y$ if and only if $y > x$.

(c) (*Order is transitive*) *If $x < y$ and $y < z$, then $x < z$.*

If $x < y$ and $y < z$ then $x - y$ and $y - z$ are both negative real numbers. Thus, summing these two negative numbers together we have $x - y + y - z = x - z$ and therefore $x < z$.

(d) (*Addition preserves order*) *If $x < y$, then $x + z < y + z$.*

If $x < y$ then $x - y$ is a negative real number. If we add a positive real number z to both x and y we have that $(x + z) - (y + z)$ so that $x + z < y + z$.

(e) (*Positive multiplication preserves order*) *If $x < y$ and z is positive, then $xz < yz$.*

Proved in the textbook.

□

5.4.3. Show that for every real number x there is exactly one integer N such that $N \leq x < N + 1$. (This integer N is called the integer part of x, and is sometimes denoted $N = \lfloor x \rfloor$.)

Proof. If x is equal to an integer let us denote it as N. Then, $N \leq x < N + 1$ is true, since $x = N$. If x is not equal to an integer let N be the closest integer to x that is less than x. Then, we must have that $N < x < N + 1$, since x is not an integer. That is, it must be between N and $N + 1$ as we chose N to be the closest integer to x that was less than it. That is, x can't more than $N+1$ as that would mean $N+1$ is less than it and closer to it than N is, which would be a contradiction. Additionally, we see that it can't be equal to $N+1$ since it is not an integer. Therefore, as stated, we have that $N < x < N+1$ and hence for every real number x we have that there is exactly one integer N such that $N \leq x < N+1$. □

5.4.4. Show that for any positive real number $x > 0$ there exists a positive integer N such that $x > 1/N > 0$.

Proof. From the Archimedean property, Corollary 5.4.13, we know that for positive real numbers x and ε that there exists a positive integer M such that $M\varepsilon > x$. Now, since x and ε are arbitrary let $x = 1$ and let us denote ε as x. Also, let us denote the positive integer M as N. Thus we have that $Nx > 1$ which it is easy to see that we have $Nx > 1 > 0$ and dividing by N we have that $x > 1/N > 0$ as desired. □

5.4.5. Prove Proposition 5.4.14. (Hint: use Exercise 5.4.4. You may also need to argue by contradiction.)

Proof. Proposition 5.4.14 claims *Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.*

If $x < y$ we know that $y - x$ is a positive real number. From Exercise 5.4.4 we also have that $y - x > 1/N > 0$ for some positive integer N. Adding x to all the sides we have that $y > 1/N + x > x$ which becomes $y > \dfrac{1+Nx}{N} > x$. From Exercise 5.4.3 we also see that $y > \dfrac{1+Nx}{N} \geq \dfrac{1+N\lfloor x \rfloor}{N} > x$. Now, $q = \dfrac{1+N\lfloor x \rfloor}{N}$ is a rational number as the numerator and denominator are both integers. Therefore, given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$. □

5.4.6. Let x, y be real numbers and let $\varepsilon > 0$ be a positive real. Show that $|x-y| < \varepsilon$ if and only if $y - \varepsilon < x < y + \varepsilon$, and that $|x - y| \leq \varepsilon$ if and only if $y - \varepsilon \leq x \leq y + \varepsilon$.

Proof. If $|x-y| < \varepsilon$ then

$$\begin{aligned}
\varepsilon - (x-y) \quad &\text{and} \quad \varepsilon - (-(x-y)) \\
\varepsilon - x + y \quad &\text{and} \quad \varepsilon - (y - x) \\
(y+\varepsilon) - x \quad &\text{and} \quad x - (y - \varepsilon) \\
x < (y+\varepsilon) \quad &\text{and} \quad (y-\varepsilon) < x
\end{aligned}$$

and therefore $y - \varepsilon < x < y + \varepsilon$. The converse argument is just going in the reverse direction.

If $|x-y| \leq \varepsilon$ then we already know the above result so for the case that $|x-y| = \varepsilon$ we have

$$\begin{aligned}
x - y = \varepsilon \quad &\text{and} \quad -(x-y) = \varepsilon \\
x = y + \varepsilon \quad &\text{and} \quad y - \varepsilon = x
\end{aligned}$$

and therefore $y - \varepsilon \leq x \leq y + \varepsilon$. The converse argument is just going in the reverse direction. □

5.4.7. Let x and y be real numbers. Show that $x \leq y + \varepsilon$ for all real numbers $\varepsilon > 0$ if and only if $x \leq y$. Show that $|x - y| \leq \varepsilon$ for all real numbers $\varepsilon > 0$ if and only if $x = y$.

Proof. If $x \leq y + \varepsilon$ for all real numbers $\varepsilon > 0$ then suppose for sake of contradiction that $x > y$. Then $x - y > 0$ is a positive number and so is $(x-y)/2$. Let $\varepsilon = (x-y)/2$. Thus, $x \leq y + \varepsilon$ which becomes $x \leq y + (x-y)/2$ and hence $2x \leq 2y + x - y$ and finally $x \leq y$, which is a contradiction. Therefore, If $x \leq y + \varepsilon$ for all real numbers $\varepsilon > 0$ then $x \leq y$.

Conversely, if $x \le y$ then suppose for the sake of contradiction that $x > y + \varepsilon$ for all real numbers $\varepsilon > 0$. Then $x - y + \varepsilon > 0$ is a positive number. But $x \le y$ and thus $x - y \le 0$, which is a not a positive number and hence a contradiction. Therefore, if $x \le y$ then $x \le y + \varepsilon$ for all $\varepsilon > 0$.

If $|x - y| \le \varepsilon$ for all real numbers $\varepsilon > 0$ then for the sake of contradiction suppose that $x \ne y$. Then we must have that $|x - y| > 0$ and thus $|x - y|/2 > 0$ as well. Let $\varepsilon = |x - y|/2$ so that $|x - y| \le \varepsilon$ becomes $|x - y| \le |x - y|/2$. Hence $2|x - y| \le |x - y|$, which is absurd as $|x - y| > 0$. Therefore, if $|x - y| \le \varepsilon$ for all real numbers $\varepsilon > 0$ then $x = y$.

Conversely, if $x = y$ then for the sake of contradiction suppose that $|x - y| > \varepsilon$ for all real numbers $\varepsilon > 0$. But $x = y$ and thus $|x - y| = 0$ and thus we have a contradiction since $\varepsilon > 0$. Therefore, if $x = y$ then $|x - y| \le \varepsilon$ for all real numbers $\varepsilon > 0$. □

5.4.8. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence of rationals, and let x be a real number. Show that if $a_n \le x$ for all $n \ge 1$, then $\mathrm{LIM}_{n\to\infty} a_n \le x$. Similarly, show that if $a_n \ge x$ for all $n \ge 1$, then $\mathrm{LIM}_{n\to\infty} a_n \ge x$. (Hint: prove by contradiction. Use Proposition 5.4.14 to find a rational between $\mathrm{LIM}_{n\to\infty} a_n$ and x, and then use Proposition 5.4.9 or Corollary 5.4.10.)

Proof. If $a_n \le x$ for all $n \ge 1$, then for the sake of contradiction suppose that $\mathrm{LIM}_{n\to\infty} a_n > x$. We know that $\mathrm{LIM}_{n\to\infty} a_n$ is a real number and since $\mathrm{LIM}_{n\to\infty} a_n > x$ we can find a rational number q such that $\mathrm{LIM}_{n\to\infty} a_n > q > x$ by Proposition 5.4.14. However, since $a_n \le x$ for all $n \ge 1$, by Corollary 5.4.10 we must have that $\mathrm{LIM}_{n\to\infty} a_n \le x$, which is a contradiction. Therefore, if $a_n \le x$ for all $n \ge 1$, then $\mathrm{LIM}_{n\to\infty} a_n \le x$.

Furthermore, if $a_n \ge x$ for all $n \ge 1$, then for the sake of contradiction suppose that $\mathrm{LIM}_{n\to\infty} a_n < x$. We know that $\mathrm{LIM}_{n\to\infty} a_n$ is a real number and since $\mathrm{LIM}_{n\to\infty} a_n < x$ we can find a rational number q such that $\mathrm{LIM}_{n\to\infty} a_n < q < x$ by Proposition 5.4.14. However, since $a_n \ge x$ for all $n \ge 1$, by Corollary 5.4.10 we must have that $\mathrm{LIM}_{n\to\infty} a_n \ge x$, which is a contradiction. Therefore, if $a_n \ge x$ for all $n \ge 1$, then $\mathrm{LIM}_{n\to\infty} a_n \ge x$. □

§5.5 The least upper bound property

5.5.1. Let E be a subset of the real numbers $\mathbf{R}$, and suppose that E has a least upper bound M which is a real number, i.e., $M = \sup(E)$. Let $-E$ be the set

$$-E := \{-x : x \in E\}$$

Show that $-M$ is the greatest lower bound of $-E$, i.e., $-M = \inf(-E)$.

Proof. Since $M = \sup(E)$ we know that $M \le U$ for all upper bounds U of E. That is, if $x \in E$ then $x \le M \le U$. Now, if we multiply this inequality on all sides by -1 we have that $-x \in -E$ such that $-x \ge -M \ge -U$ for all lower bounds $-U$. Therefore, $-M$ is the greatest lower bound of $-E$, i.e., $-M = \inf(-E)$. □

5.5.2. Let E be a non-empty subset of $\mathbf{R}$, let $n \ge 1$ be an integer, and let $L < K$ be integers. Suppose that K/n is an upper bound for E, but that L/n is not an upper bound for E. Without using Theorem 5.5.9, show that there exists an integer $L < m \le K$ such that m/n is an upper bound for E, but that $(m-1)/n$ is not an upper bound for E. (Hint: prove by contradiction, and use induction. It may also help to draw a picture of the situation.)

Proof. For the sake of contradiction let us assume that for some $n \geq 1$ that there does not exist an integer $L < m \leq K$ such that m/n is an upper bound for E, but that $(m-1)/n$ is not an upper bound for E. Let k be a positive integer where this is the case so that an integer m does not exist such that m/k is an upper bound for E, but that $(m-1)/k$ is not an upper bound for E. Since $L < K$ and L/n is not an upper bound for E and K/n is an upper bound for E for $n \geq 1$ by hypothesis, we see that L/k is not an upper bound for E and that K/k is an upper bound for E. Now, since L and K are arbitrary, let $m = K$ and $m - 1 = L$ so that $L < m \leq K$ becomes true. Obviously $m/k = K/k$ is an upper bound for E and also $(m-1)/k = L/k$ is not an upper bound for E, which is a contradiction that m did not exist. Therefore, we conclude that there exists an integer $L < m \leq K$ such that m/n is an upper bound for E, but that $(m-1)/n$ is not an upper bound for E for any $n \geq 1$. □

5.5.3. Let E be a non-empty subset of $\mathbf{R}$, let $n \geq 1$ be an integer, and let m, m' be integers with the properties that m/n and m'/n are upper bounds for E, but $(m-1)/n$ and $(m'-1)/n$ are not upper bounds for E. Show that $m = m'$. This shows that the integer m constructed in Exercise 5.5.2 is unique. (Hint: again, drawing a picture will be helpful.)

Proof. For the sake of contradiction let us suppose that $m \neq m'$. Then either $m > m'$ or $m < m'$. If $m > m'$ then $m/n > m'/n$. But this means that $(m-1)/n \geq m'/n$ showing that $(m-1)/n$ is an upper bound, a contradiction. Similarly, if $m < m'$ then $m/n < m'/n$. But this means that $m/n \leq (m'-1)/n$ showing that $(m'-1)/n$ is an upper bound, a contradiction. Therefore, $m = m'$. □

5.5.4. Let $q_1, q_2, q_3, \ldots$ be a sequence of rational numbers with the property that $|q_n - q_{n'}| \leq \frac{1}{M}$ whenever $M \geq 1$ is an integer and $n, n' \geq M$. Show that $q_1, q_2, q_3, \ldots$ is a Cauchy sequence. Furthermore, if $S := \text{LIM}_{n \to \infty} q_n$ show that $|q_M - S| \leq \frac{1}{M}$ for every $M \geq 1$. (Hint: use Exercise 5.4.8.)

Proof. To show that $(q_n)_{n=1}^{\infty}$ is a Cauchy sequence we need to show that it is eventually ε-close for all $\varepsilon > 0$. Since $M \geq 1$ let $\varepsilon' = 1/M > 0$. Then since we have the property $|q_n - q_{n'}| \leq 1/M = \varepsilon'$ for $n, n' \geq M$, we see that this sequence is eventually ε'-close. Now, since $0 < \varepsilon' \leq 1$ we see that this sequence is also eventually ε-close for any $0 < \varepsilon' \leq 1 < \varepsilon$. Therefore, $(q_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Furthermore, if $S := \text{LIM}_{n \to \infty} q_n$ then we know that S is a real number by Definition 5.3.1. From Exercise 5.4.8 we then know that either $q_M \leq S$ or $q_M \geq S$ for $M \geq 1$. That is, each element q_M of the Cauchy sequence $(q_n)_{n=1}^{\infty}$, is either less than or equal to S or greater than or equal to S. Therefore, □

5.5.5. Establish an analogue of Proposition 5.4.14, in which "rational" is replaced by "irrational".

Proof. We need to show that we can find an irrational number between any two real numbers.

If $x < y$ then by Proposition 5.5.12 we know that there exists a positive real number, which is not rational, z such that $z^2 = 2$. Then $z + x < z + y$ and since $z + x$ and $z + y$ are both real numbers we know there exists a rational number q such that $z + x < q < z + y$. Thus, we have that $x < q - z < y$ and therefore, between any two real numbers we can find an irrational number. (Note: we have not used the notation $\sqrt{2}$ as this hasn't been introduced yet but will appear in the next section.) □

§5.6 Real exponentiation, part I

5.6.1. Prove Lemma 5.6.6. (Hints: review the proof of Proposition 5.5.12. Also, you will find proof by contradiction a useful tool, especially when combined with the trichotomy of order in Proposition 5.4.7 and

Proposition 5.4.12. The earlier parts of the lemma can be used to prove later parts of the lemma. With part (e), first show that if $x > 1$ then $x^{1/n} > 1$, and if $x < 1$ then $x^{1/n} < 1$.)

Proof. Lemma 5.6.6. *Let $x, y \geq 0$ be non-negative reals, and let $n, m \geq 1$ be positive integers.*

(a) *If $y = x^{1/n}$, then $y^n = x$.*

Let E be the set $\{y \in \mathbf{R} : y \geq 0 \text{ and } y^n \leq x\}$ so that $x^{1/n} := \sup E$. We will show that $y^n = x$. We argue this by contradiction. We show that both $y^n < x$ and $y^n > x$ lead to contradictions. First suppose that $y^n < x$. Let $0 < \varepsilon < 1$ be a small number; then we have

$$\begin{aligned}(y+\varepsilon)^n &= \binom{n}{0}y^n\varepsilon^0 + \binom{n}{1}y^{n-1}\varepsilon^1 + \binom{n}{2}y^{n-2}\varepsilon^2 + \cdots + \binom{n}{n-1}y^1\varepsilon^{n-1} + \binom{n}{n}y^0\varepsilon^n \\ &= y^n + \left[\binom{n}{1}y^{n-1} + \binom{n}{2}y^{n-2}\varepsilon + \cdots + \binom{n}{n-1}y^1\varepsilon^{n-2} + \binom{n}{n}y^0\varepsilon^{n-1}\right]\varepsilon\end{aligned}$$

Let Z denote the value in the square brackets above. Since $y^n < x$ we can choose an $0 < \varepsilon < 1$ such that $y^n + Z\varepsilon < x$, thus $(y+\varepsilon)^n < x$. However, by hypothesis $y = x^{1/n}$ so therefore $(x^{1/n} + \varepsilon)^n < x$ and by construction of E this means that $x^{1/n} + \varepsilon \in E$. This contradicts the fact that $x^{1/n}$ is an upper bound of E.

Now suppose that $y^n > x$. Let $0 < \varepsilon < 1$ be a small number; then we have

$$\begin{aligned}(y-\varepsilon)^n &= \binom{n}{0}y^n\varepsilon^0 - \binom{n}{1}y^{n-1}\varepsilon^1 + \binom{n}{2}y^{n-2}\varepsilon^2 - \cdots \pm \binom{n}{n-1}y^1\varepsilon^{n-1} \mp \binom{n}{n}y^0\varepsilon^n \\ &= y^n - \left[\binom{n}{1}y^{n-1} + \binom{n}{2}y^{n-2}\varepsilon - \cdots \pm \binom{n}{n-1}y^1\varepsilon^{n-2} \mp \binom{n}{n}y^0\varepsilon^{n-1}\right]\varepsilon\end{aligned}$$

(Note: $\pm$ and $\mp$ denote the signs in the binomial expansion. If n is odd, use the top signs while if n is even use the bottom signs. This actually doesn't make a difference in the analysis below.)

Again, let Z denote the value in the square brackets above. Since $y^n > x$ we can choose an $0 < \varepsilon < 1$ such that $y^n - Z\varepsilon > x$, thus $(y-\varepsilon)^n > x$. However, by hypothesis $y = x^{1/n}$ so therefore $(x^{1/n} - \varepsilon)^n > x$. But then this implies that $x^{1/n} - \varepsilon \geq x$ for all $y \in E$. Thus $x^{1/n} - \varepsilon$ is an upper bound for E, which contradicts the fact that $x^{1/n}$ is the *least* upper bound of E. From these two contradictions we see that $y^n = x$, as desired.

(b) *Conversely, if $y^n = x$, then $y = x^{1/n}$.*

Let E be the set $\{y \in \mathbf{R} : y \geq 0 \text{ and } y^n \leq x\}$ so that $x^{1/n} := \sup E$. We will show that $y = x^{1/n}$. We argue this by contradiction. We show that both $y < x^{1/n}$ and $y > x^{1/n}$ lead to contradictions. First suppose that $y < x^{1/n}$. Now, since $y^n = x$ by Definition 5.6.4 we must have that $y \in E$. We will show that it is larger than or equal to any other element of E and therefore is an upper bound. Let $y' \in E$ such that $y' \neq y$ and for sake of contradiction suppose that $y' > y$. Then, by Proposition 4.3.12(b) we must have that ${y'}^n > y^n = x$. But this means that $y' \notin E$, a contradiction. Therefore, we must have that $y' \leq y$ and therefore y is an upper bound of E. But $y < x^{1/n}$ so therefore $x^{1/n}$ is not $\sup E$, a contradiction. Now suppose that $y > x^{1/n}$. Now, since $y^n = x$ by Definition 5.6.4 we must have that $y \in E$. But if $y > x^{1/n}$ this means that $x^{1/n}$ is not an upper bound for E, a contradiction since $x^{1/n} = \sup E$. From these two contradictions we see that $y = x^{1/n}$, as desired.

(c) *$x^{1/n}$ is a positive real number.*

We are given that $x \geq 0$ and that $n \geq 1$. If $x = 0$ then $x^{1/n} = 0$ which is a non-negative real number and not a positive real number so I am not entirely sure why Tao has this part of the Lemma stated as such. Let us assume that $x > 0$. Then, from Definition 5.6.4 we must have that $x^{1/n}$ is a positive real number as it is the supremum of the set $E = \{y \in \mathbf{R} : y \geq 0 \text{ and } y^n \leq x\}$ and as we can see, since $x > 0$, there will be some positive real numbers in this set and therefore the supremum of E must be some positive real number.

(d) *We have $x > y$ if and only if $x^{1/n} > y^{1/n}$.*

Let $E_1 = \{y' \in \mathbf{R} : y' \geq 0 \text{ and } y'^n \leq x\}$ and $E_2 = \{y'' \in \mathbf{R} : y'' \geq 0 \text{ and } y''^n \leq y\}$.

If $x > y$ then there exists $y' \in E_1$ such that $y' \notin E_2$ and therefore $E_2 \subset E_1$. Thus, $\sup E_2 < \sup E_1$ and hence $y^{1/n} < x^{1/n}$. Conversely, if $x^{1/n} > y^{1/n}$ then $\sup E_1 > \sup E_2$ and therefore $E_2 \subset E_1$. Thus, there exists $y' \in E_1$ such that $y' \notin E_2$ and hence $x > y$.

(e) *If $x > 1$, then $x^{1/k}$ is a decreasing function of k. If $x < 1$, then $x^{1/k}$ is an increasing function of k. If $x = 1$, then $x^{1/k} = 1$ for all k.*

First, note that

1. If $x > 1$ then for $n \geq 1$, we have that $x^{1/n} > 1^{1/n} = 1$.

2. If $x < 1$ then for $n \geq 1$, we have that $x^{1/n} < 1^{1/n} = 1$.

If $x > 1$, as k increases the values of y must decrease to make it so that $y^k \leq x$. Thus, $\sup E$ will be smaller as k grows and therefore $x^{1/k}$ is a *decreasing* function of k.

If $x < 1$, as k increases the values of y must increase to make it so that $y^k \leq x$. This may seem counter-intuitive but we must remember that $0 \leq y < 1$. Thus, $\sup E$ will be larger as k grows and therefore $x^{1/k}$ is an *increasing* function of k.

If $x = 1$ then, since exponentiation is just repeated multiplication, we must have that 1 to any power is equal to 1 since 1 multiplied by itself no matter how many times, is still 1. Thus, when $x = 1$, $x^{1/k} = 1^{1/k} = 1$ for all k.

(f) *We have $(xy)^{1/n} = x^{1/n}y^{1/n}$.*

Let $E_1 = \{y' \in \mathbf{R} : y' \geq 0 \text{ and } y'^n \leq x\}$ and $E_2 = \{y'' \in \mathbf{R} : y'' \geq 0 \text{ and } y''^n \leq y\}$.

Since $y'^n \leq x$ and $y''^n \leq y$ then $y'^n y \leq xy$ from which we then see that $y'^n y''^n \leq y'^n y \leq xy$ and therefore $(y'y'')^n \leq xy$. Let $E = \{y'y'' \in \mathbf{R} : y'y'' \geq 0 \text{ and } (y'y'')^n \leq xy\}$.

For the sake of contradiction, suppose that $(xy)^{1/n} > x^{1/n}y^{1/n}$. Now, if $y' \in E_1$ then $y' \leq (\sup E_1 = x^{1/n})$ and if $y'' \in E_2$ then $y'' \leq (\sup E_2 = y^{1/n})$. Since $y', y'' \geq 0$ starting from $y' \leq x^{1/n}$ we must have that $y'y'' \leq x^{1/n}y'' \leq x^{1/n}y^{1/n}$ and therefore $y'y'' \leq x^{1/n}y^{1/n}$ which shows that $x^{1/n}y^{1/n}$ is an upper bound for E, which is a contradiction as $(xy)^{1/n}$ is the supremum of E. Conversely, suppose that $(xy)^{1/n} < x^{1/n}y^{1/n}$. As we saw, any element $y'y''$ of E has the property that $y'y'' \leq x^{1/n}y^{1/n}$. Since $(xy)^{1/n} < x^{1/n}y^{1/n}$ we must have that $x^{1/n}y^{1/n} - (xy)^{1/n} > 0$. Let us denote this positive number as ε. Then we must have that $(xy)^{1/n} + \varepsilon \leq x^{1/n}y^{1/n}$, and therefore $((xy)^{1/n} + \varepsilon) \in E$, a contradiction as this shows that $(xy)^{1/n}$ is not an upper bound of E. From these two contradictions we see that $(xy)^{1/n} = x^{1/n}y^{1/n}$, as desired.

(g) *We have $(x^{1/n})^{1/m} = x^{1/nm}$.*

Let $E_1 = \{y \in \mathbf{R} : y \geq 0 \text{ and } y^m \leq x^{1/n}\}$ and $E_2 = \{y \in \mathbf{R} : y \geq 0 \text{ and } y^{nm} \leq x\}$. Thus, $(x^{1/n})^{1/m} = \sup E_1$ and $x^{1/nm} = \sup E_2$. Now we will show that $E_1 = E_2$ and therefore $(x^{1/n})^{1/m} = x^{1/nm}$.

To do this, we will simply show that the inequality expressions that define these sets are equal which means the sets are equal, which will give us the result desired. That is, we need to show that $y^m \le x^{1/n}$ and $y^{nm} \le x$ are the same. For $y^m \le x^{1/n}$, we must have that $(y^m)^n \le (x^{1/n})^n$. Now, let $u = x^{1/n}$ and from part (a) of this Lemma we then know that $u^n = x$. Thus, $(y^m)^n \le x$ and from Proposition 5.6.3 we know that $(y^m)^n = y^{mn} = y^{nm}$ so that we have $y^{nm} \le x$. Therefore, $E_1 = E_2$ showing that $(x^{1/n})^{1/m} = x^{1/nm}$.

□

5.6.2. Prove Lemma 5.6.9. (Hint: you should rely mainly on Lemma 5.6.6 and on algebra.)

Proof. Lemma 5.6.9. *Let $x, y > 0$ be positive reals, and let q, r be rationals.*

Let $q = a/b$ and $r = c/d$ with a, c being integers and b, d being positive integers.

(a) x^1 *is a positive real.*

From Lemma 5.6.6(c) since $1/1 = 1$ we have that x^1 is a positive real number.

(b) $x^{q+r} = x^q x^r$ *and* $(x^q)^r = x^{qr}$.

$$\begin{aligned}
x^{q+r} &= x^{a/b+c/d} \\
&= x^{(ad+bc)/bd} \\
&= (x^{1/bd})^{(ad+bc)} && \text{[Def. 5.6.7]} \\
&= x^{ad/bd} x^{bc/bd} && \text{[Prop. 5.6.3]} \\
&= x^{a/b} x^{c/d} \\
&= x^q x^r
\end{aligned}$$

$$\begin{aligned}
(x^q)^r &= ((x^{1/b})^a)^{c/d} && \text{[Def. 5.6.7]} \\
&= (((x^{1/b})^a)^{1/d})^c && \text{[Def. 5.6.7]} \\
&= (((x^{-b})^a)^{-d})^c && \text{[Def. 5.6.2]} \\
&= x^{(-b)a(-d)c} && \text{[Prop. 4.3.12(a)]} \\
&= x^{ac/bd} && \text{[Def. 5.6.2]} \\
&= x^{qr}
\end{aligned}$$

(c) $x^{-q} = 1/x^q$.

$$\begin{aligned}
x^{-q} &= x^{-(a/b)} \\
&= (x^{1/b})^{-a} && \text{[Def. 5.6.7]} \\
&= 1/(x^{1/b})^a && \text{[Def. 5.6.2]} \\
&= 1/x^q
\end{aligned}$$

(d) *If $q > 0$, then $x > y$ if and only if $x^q > y^q$.*

If $x > y$ then

$$x > y$$

$$x^{1/b} > y^{1/b} \quad \text{[Lemma 5.6.6(d)]}$$

Let $X = x^{1/b}$ and $Y = y^{1/b}$ so that

$$\begin{aligned} X &> Y \\ X^a &> Y^a && \text{[Prop. 4.3.12(b)]} \\ (x^{1/b})^a &> (y^{1/b})^a \\ x^q &> y^q && \text{[Def. 5.6.7]} \end{aligned}$$

The converse argument is the reverse of the above.

(e) If $x > 1$, then $x^q > x^r$ if and only if $q > r$. If $x < 1$, then $x^q > x^r$ if and only if $q < r$.

If $x > 1$ and $q > r$ then

$$\begin{aligned} q &> r \\ a/b &> c/d \\ ad &> bc \\ x^{ad} &> x^{bc} && \text{[}x > 1 \text{ and Def. 4.3.9]} \\ (x^a)^d &> (x^c)^b && \text{[Prop. 4.3.12(a)]} \end{aligned}$$

Let $X_1 = (x^a)^d$ and $X_2 = (x^c)^b$ so that

$$\begin{aligned} X_1 &> X_2 \\ (X_1)^{1/bd} &> (X_2)^{1/bd} && \text{[Lemma 5.6.6(d)]} \\ (x^a)^{1/b} &> (x^c)^{1/d} \\ (x^{1/b})^a &> (x^{1/d})^c && \text{[Lemma 5.6.9(b)]} \\ x^q &> x^r \end{aligned}$$

The converse argument is the reverse of the above.

If $x < 1$ and $q < r$ then

$$\begin{aligned} q &< r \\ a/b &< c/d \\ ad &< bc \\ x^{ad} &> x^{bc} && \text{[}x < 1 \text{ and Def. 4.3.9]} \\ (x^a)^d &> (x^c)^b && \text{[Prop. 4.3.12(a)]} \end{aligned}$$

Let $X_1 = (x^a)^d$ and $X_2 = (x^c)^b$ so that

$$\begin{aligned} X_1 &> X_2 \\ (X_1)^{1/bd} &> (X_2)^{1/bd} && \text{[Lemma 5.6.6(d)]} \\ (x^a)^{1/b} &> (x^c)^{1/d} \\ (x^{1/b})^a &> (x^{1/d})^c && \text{[Lemma 5.6.9(b)]} \end{aligned}$$

$$x^q > x^r$$

The converse argument is the reverse of the above.

□

5.6.3. If x is a real number, show that $|x| = \left(x^2\right)^{1/2}$.

Proof. If x is a real number then by definition $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$. For non-negative x we obviously have that $(x^2)^{1/2} = x$. For negative x we have that $((-x)^2)^{1/2} = ((-x)(-x))^{1/2} = (x^2)^{1/2} = x$. Therefore, $|x| = (x^2)^{1/2}$. □