Analysis I

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Chapter A - Appendix: the basics of mathematical logic

Exercises:

§A.1 Mathematical statements

A.1.1. What is the negation of the statement "either X is true, or Y is true, but not both"?

The negation of this statement is: Neither X nor Y is true, or both X and Y are true.

A.1.2. What is the negation of the statement "X is true if and only if Y is true"? (There may be multiple ways to phrase this negation).

The negation of this statement is: X is true and Y is not true, or Y is true and X is not true.

A.1.3. Suppose that you have shown that whenever X is true, then Y is true, and whenever X is false, then Y is false. Have you now demonstrated that X and Y are logically equivalent? Explain.

Yes, this demonstrates that X and Y are logically equivalent since X is true if and only if Y is true because we can't have the situation where Y is true and X is false since when X is false Y must also be false. Similarly, X is false if and only if Y is false because we can't have the situation where Y is false and X is true since when X is true Y must also be true. In other words, they are both true or both false and therefore they are logically equivalent.

A.1.4. Suppose that you have shown that whenever X is true, then Y is true, and whenever Y is false, then X is false. Have you now demonstrated that X is true if and only if Y is true? Explain.

No, because we can have the situation where X is false and Y is true. The hypotheses don't constrain the truth or falsity of Y based on the falsity of X.

A.1.5. Suppose you know that X is true if and only if Y is true, and you know that Y is true if and only if Z is true. Is this enough to show that X, Y, Z are all logically equivalent? Explain.

Yes, because if X implies Y and Y implies Z then X implies Z and if Z implies Y and Y implies X then Z implies X. Therefore, $X \iff Z$ and this, with $X \iff Y$ and $Y \iff Z$, shows that X, Y, Z are all logically equivalent.

A.1.6. Suppose you know that whenever X is true, then Y is true; that whenever Y is true, then Z is true; and whenever Z is true, then X is true. Is this enough to show that X, Y, Z are all logically equivalent? Explain.

Yes because we basically have a cycle of truth. That is, if any of X, Y, Z are true, then the others must logically be as well. That is, they are either all true or all false and therefore they are logically equivalent.

§A.2 Implication

No exercises for section A.2.

§A.3 The structure of proofs

No exercises for section A.3.

§A.4 Variables and quantifiers

No exercises for section A.4.

§A.5 Nested quantifiers

A.5.1. What does each of the following statements mean, and which of them are true? Can you find gaming metaphors for each of these statements?

(a) For every positive number x, and every positive number y, we have $y^2 = x$.

This statement means that for all positive numbers x and y we have $y^2 = x$, which is obviously false, as for example with $x = 2 > 0$ and $y = 3 > 0$ we have that $3^2 = 2$, which is false.

Gaming metaphor is your opponent would pick x and y for you and you would need to show that $y^2 = x$ is true regardless of the x and y chosen by your opponent.

(b) There exists a positive number x such that for every positive number y, we have $y^2 = x$.

This statement means that for some x and for any y, we have $y^2 = x$. This x doesn't exist so the statement is false. We can prove this via contradiction. For the sake of contradiction suppose that there is some x such that for any y, we have $y^2 = x$. Then, let us take y_1 and y_2 where $y_1 \neq y_2$ which will then give us that $y_1^2 = x$ and $y_2^2 = x$ so that $y_1 = \sqrt{x}$ and $y_2 = \sqrt{x}$. But this is a contradiction since $y_1 \neq y_2$. Therefore, we must have that x doesn't exist such that for any y, we have $y^2 = x$.

Gaming metaphor is you can choose x but your opponent then chooses any y and you have to show that $y^2 = x$.

(c) There exists a positive number x, and there exists a positive number y, such that $y^2 = x$.

This statement means that for some x and for some y, then we have $y^2 = x$. This is as true statement. For example $x = 4$ and $y = 2$ suffice.

Gaming metaphor is you can choose x and y to show that $y^2 = x$.

(d) For every positive number y, there exists a positive number x such that $y^2 = x$.

This statement means that for any positive number y and for some positive number x we have $y^2 = x$. This is a true statement as we can just pick whatever x is equal to y^2.

Gaming metaphor is that your opponent picks y and you can pick x to show $y^2 = x$.

(e) There exists a positive number y such that for every positive number x we have $y^2 = x$.

This statement means that for some positive number y and for every positive number x we have $y^2 = x$. This is a false statement and proof would be similar to (b) above (we will leave this to the reader).

Gaming metaphor is that you can pick y but your opponent picks x and you have to show that $y^2 = x$.

§A.6 Some examples of proofs and quantifiers

No exercises for section A.6.

§A.7 Equality

A.7.1. Suppose you have four real numbers a, b, c, d and you know that $a = b$ and $c = d$. Use the above four axioms to deduce that $a + d = b + c$.

The four axioms of equality again are:

- (Reflexive axiom). Given any object x, we have $x = x$.
- (Symmetry axiom). Given any two objects x and y of the same type, if $x = y$, then $y = x$.
- (Transitive axiom). Given any three objects x, y, z of the same type, if $x = y$ and $y = z$, then $x = z$.
- (Substitution axiom). Given any two objects x and y of the same type, if $x = y$, then $f(x) = f(y)$ for all functions or operations f. Similarly, for any property $P(x)$ depending on x, if $x = y$, then $P(x)$ and $P(y)$ are equivalent statements.

Now we have four real numbers a, b, c, and d, which are all of the same type. Then

$$\begin{aligned} a &= b && \text{[hypothesis]} \\ a + c &= b + c && \text{[Substitution - } f(a,c) = f(b,c)\text{, where } f \text{ is addition]} \\ a + d &= b + c && \text{[Substitution - } c = d\text{]} \end{aligned}$$

We really only needed the Substitution axiom but you can notice that at each step of the deduction all the axioms still hold (if applicable) for the statement as it progresses to its final form. Furthermore, the hypotheses themselves relied on the axioms of equality and because of this, in essence, we built the deduction on top of these axioms (i.e., even though we didn't show how all the axioms of equality came into "play" for each step of the deduction they were all present on some level due to the hypotheses being equalities.)