Principles of Mathematical Analysis, 3rd Edition

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Chapter 1 - THE REAL AND COMPLEX NUMBER SYSTEMS

Exercises:

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. If r is *rational* ($r \neq 0$) and x is *irrational*, prove that $r + x$ and rx are irrational.

Proof. Let $r = m/n$ for integers m and n.

If $r + x$ is rational, then for integers q and p.

$$\begin{aligned} r + x &= p/q \\ x = &= p/q - m/n \\ &= (pn - mq)/(qn) \end{aligned}$$

which is a rational number and a contradiction.

Furthermore, if rx is rational, then

$$\begin{aligned} rx &= p/q \\ (m/n)x &= p/q \\ x &= (pn)/(mq) \end{aligned}$$

which is a rational number and a contradiction.

Therefore, if r is rational ($r \neq 0$) and x is irrational, then $r + x$ and rx are irrational. □

2. Prove that there is no rational number whose square is 12.

Proof. If $(m/n)^2 = 12$, then for integers m and n that do not have a common factor

$$\begin{aligned} m^2/n^2 &= 12 \\ m^2/n^2 &= 3 \cdot 4 \\ m^2 &= 3 \cdot 4 \cdot n^2 \end{aligned}$$

3 is a factor of m^2 and therefore m. If $m = (3r)$ with integer r, then

$$\begin{aligned} (3r)^2 &= 3 \cdot 4 \cdot n^2 \\ 3r^2 &= 4 \cdot n^2 \end{aligned}$$

3 is also a factor of n^2 and therefore n, which is a contradiction.

Therefore, there is no rational number whose square is 12. □

3. Prove proposition 1.15.

Proof.

(a) If $x \neq 0$ and $xy = xz$ then $y = z$.

$$\begin{aligned} xy &= xz \\ (1/x)(xy) &= (1/x)(xz) \\ y &= z \end{aligned}$$

(b) If $x \neq 0$ and $xy = x$ then $y = 1$.

$$\begin{aligned} xy &= x \\ (1/x)(xy) &= (1/x)x \\ y &= 1 \end{aligned}$$

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.

$$\begin{aligned} xy &= 1 \\ (1/x)(xy) &= (1/x) \\ y &= (1/x) \end{aligned}$$

(d) If $x \neq 0$ then $1/(1/x) = x$.

$$\begin{aligned} 1/(1/x) &= 1 \cdot x/1 \\ &= x \end{aligned}$$

□

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound on E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. If $\alpha \leq x, \forall x \in E$ and $x \leq \beta, \forall x \in E$, then $\alpha \leq x \leq \beta \implies \alpha \leq \beta$. □

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Proof. If $\alpha = \inf A$, then $\alpha \leq x, \forall x \in A$. If $\beta = -\alpha$, then

$$\begin{aligned} -\alpha &\geq -x, \ \forall(-x) \in -A \\ \beta &\geq -x, \ \forall(-x) \in -A \end{aligned}$$

Since α was the infimum of A, β is the supremum for $-A$. Therefore, since $\alpha = \inf A$ and $\beta = \sup(-A)$ we have that

$$\begin{aligned} -\alpha &= \beta \\ -\inf A &= \sup(-A) \\ \inf A &= -\sup(-A). \end{aligned}$$

□

6. Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}$$

Proof.

$$\begin{aligned}(b^m)^{1/n} &= b^{m/n} \\ &= b^r \\ &= b^{p/q} \\ &= (b^p)^{1/q}\end{aligned}$$

Hence it makes sense to define

$$b^r = (b^m)^{1/n}$$

□

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

Proof. If $r = m/n$ and $s = p/q$, then

$$\begin{aligned}b^{r+s} &= b^{m/n+p/q} \\ &= b^{(mq+pn)/nq} \\ &= (b^{mq+pn})^{1/nq} \\ &= (b^{mq} b^{pn})^{1/nq} && \text{[law of exponents for integer exponents]} \\ &= (b^{mq})^{1/nq} (b^{pn})^{1/nq} && \text{[Corollary of Theorem 1.21]} \\ &= (b^{mq/nq})(b^{pn/nq}) \\ &= b^{m/n} b^{p/n} \\ &= b^r b^s\end{aligned}$$

□

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational.

Proof. When r is rational and $t \leq r$ we have that $b^t \leq b^r$ and therefore b^r is an upper bound for $B(r)$. Additionally, $b^r \in B(r)$ so that $b^r \leq U$, for any upper bound U. Therefore, $b^r = \sup B(r)$ when r is rational.

Hence it makes sense to define

$$b^x = \sup B(x)$$

for ever real x. □

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Proof. From part (c) we know that $b^x = \sup B(x)$, therefore we see that $b^x b^y = \sup B(x) \sup B(y)$. Now we need to show that $b^{x+y} = \sup B(x) \sup B(y)$.

If $b^t \in B(x+y)$, then $t \leq x+y$. Let $t = r+s$ with $r \leq x$ and $s \leq y$ so that

$$\begin{aligned} b^t &= b^{r+s} \\ &= b^r b^s \\ &\leq \sup B(x) \sup B(y) \end{aligned}$$

and therefore $\sup B(x) \sup B(y)$ is an upper bound for $B(x+y)$.

Now we will show that it is also the least upper bound. Let $n \in B(x+y)$ and $0 < n < \sup B(x) \sup B(y)$. Then $0 < \dfrac{n}{\sup B(x)} < \sup B(y)$. Let $m = \dfrac{1}{2}\left[\dfrac{n}{\sup B(x)} + \sup B(y)\right]$ so that $\dfrac{n}{\sup B(x)} < m < \sup B(y)$.

Therefore, since

$$\frac{n}{\sup B(x)} < m \implies \frac{n}{m} < \sup B(x)$$

Now, with the same argument we used for m above, we know that there exists $u \in B(x)$ such that $\dfrac{n}{m} < u < \sup B(x)$. Again, using the same argument, since $m < \sup B(y)$, there exists $v \in B(y)$ such that $m < v < \sup B(y)$. This shows us that

$$n = \frac{n}{m} \cdot m < u \cdot v < \sup B(x) \sup B(y)$$

and since $uv \in B(x+y)$, we see that n is not an upper bound and thus $\sup B(x) \sup B(y)$ is the least upper bound of $B(x+y)$.

Therefore, $b^{x+y} = b^x b^y$. □

7. Fix $b > 1, y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b*.)

(a) For any positive integer n, $b^n - 1 \geq n(b-1)$.

Proof.
base case: $n = 1,\ b^1 - 1 \geq 1(b-1)$.

induction hypothesis: $n = k,\ b^k - 1 \geq k(b-1)$.

induction step: Let $n = k+1$. Then

$$\begin{aligned} b^{k+1} - 1 &\geq (k+1)(b-1) \\ bb^k - 1 &\geq kb - k + b - 1 \\ b^{k+1} - b &\geq k(b-1) \\ b(b^k - 1) &\geq k(b-1) \end{aligned}$$

which holds due to induction hypothesis and $b > 1$. □

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

Proof. Since

$$\begin{aligned} b &> 1 \\ (b)^{1/n} &> (1)^{1/n} \\ b^{1/n} &> 1 \end{aligned}$$

we see that we can swap b for $b^{1/n}$ as the criteria is still met from part (a), and therefore $b - 1 \geq n(b^{1/n} - 1)$. ∎

(c) If $t > 1$ and $n > (b-1)(t-1)$, then $b^{1/n} < t$.

Proof. From part (b),

$$\begin{aligned} b - 1 &\geq n(b^{1/n} - 1) \\ b - 1 &> (b-1)(t-1)(b^{1/n} - 1) \\ \frac{1}{(b-1)}(b-1) &> \frac{1}{(b-1)}(b-1)(t-1)(b^{1/n} - 1) \\ 1 &> (t-1)(b^{1/n} - 1) \\ 1 &> tb^{1/n} - t - b^{1/n} + 1 \\ t &> b^{1/n}(t-1) \\ t/(t-1) &> b^{1/n} \end{aligned}$$

If $t > 1$, then $t > t/(t-1) > b^{1/n}$. ∎

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this, apply part (c) with $t = yb^{-w}$.

Proof. From part (c),

$$\begin{aligned} b^{1/n} &< t \\ \implies b^{1/n} &< yb^{-w} \\ \implies b^w(b^{1/n}) &< yb^w(b^{-w}) \\ \implies b^{w+1/n} &< y \end{aligned}$$

∎

(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

Proof. Same as proof for (d) but with $t = y^{-1}b^w$. ∎

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

Proof. If $x = \sup A$, then there are three possible cases: $b^x < y$, $b^x > y$, or $b^x = y$.

If $b^x < y$, then $x \in A$ and from part (d) $x + (1/n) \in A$ for sufficiently large n. This contradicts the fact that x is an upper bound for A.

If $b^x > y$, then from part (e) $x - (1/n)$ for sufficiently large n. This contradicts the fact that x is the least upper bound.

The only possibility left is that $b^x = y$. ∎

(g) Prove that this x is unique.

Proof. If s satisfies the above properties, then

$$\begin{aligned} b^s &= y = b^x \\ b^{-s}b^s &= b^x b^{-s} \end{aligned}$$

$$
\begin{aligned}
1 &= b^x b^{-s} \\
1 &= b^{x-s} \\
\implies x - s &= 0 \\
\implies x &= s
\end{aligned}
$$

□

8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint*: -1 is a square.

Proof. From definition 1.17 (ii) for an ordered field F: $xy > 0$ if $x \in F$, $y \in F$, $x > 0$, $y > 0$.

For the complex field, if $x = y = i$, then $xy = i^2 = -1 \not> 0$. □

9. Suppose $z = a + bi, w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relations is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

Proof. With the given constraints, z and w can have the cases:

$$
\begin{aligned}
a < c,\ b = d &\implies z < w \\
a < c,\ b < d &\implies z < w \\
a < c,\ b > d &\implies z < w \\
a > c,\ b = d &\implies z > w \\
a > c,\ b < d &\implies z > w \\
a > c,\ b > d &\implies z > w \\
a = c,\ b = d &\implies z = w \\
a = c,\ b < d &\implies z < w \\
a = c,\ b > d &\implies z > w
\end{aligned}
$$

We know that the real numbers is an ordered set so that $a, b, c, d \in \mathbb{R}$ satisfy conditions (i) and (ii) of Definition 1.5. From the above cases we can see that z and w satisify (i). We will need to check that they meet (ii):

Let $t = e + fi$.

Cases ($a < c, a > c$): Let $z < w$ and $w < t$. Then we know that $a < c$ and $c < e$, where b, d, f can be any value. Since the real numbers are transitive, we see that $a < c, c < e \implies a < e$. Therefore, $z < w, w < t \implies z < t$. It is easy to see that the same reasoning applies for all the cases with $a > c$.

Cases ($a = c$): For the case that $a = c$ and $b = d$ we have the case of equality $z = w$. Obviously if $z = w$ and $w = t$ we have that $z = t$.

For the other cases in this group we can test them in tandem. For the case $a = c, b < d \implies z < w$ we can use z and w as is while for the case $a = c, b > d \implies z > w$ let us use t in place for z and leave w as is. Then we get that $a = c, b < d, e = c, f > d \implies a = c = e, b < d < f \implies a = e, b < f$. Therefore, we see that $z < w, w < t \implies z < t$.

This ordered set does not have the least-upper-bound property. To see this, let $B = \{(0, b) \mid b \in \mathbb{R}\}$ be a subset of our ordered set. This subset has an upper bound, since $(a, 0) > (0, b)$ for any $a > 0$. However, for any proposed least upper bound $(a, b), a > 0$, we see that

$$(a,b) > \left(\frac{a}{2}, b\right) > (0,b)$$

Where $\left(\frac{a}{2}, b\right)$ is an upper bound less than the proposed least upper bound, a contradiction. □

10. Suppose $z = a + bi, w = u + vi$, and

$$a = ((|w| + u)/2)^{1/2}, \;\; b = ((|w| - u)/2)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $\overline{z}^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Proof.

$$\begin{aligned} z^2 &= (a+bi)(a+bi) \\ &= a^2 + 2abi - b^2 \\ &= \frac{|w|+u}{2} + 2i\left[\left(\frac{|w|+u}{2}\right)^{1/2}\left(\frac{|w|-u}{2}\right)^{1/2}\right] - \frac{|w|-u}{2} \\ &= u + ((|w|+u)(|w|-u))^{1/2} i \\ &= u + vi \end{aligned}$$

Note that here we used the fact that $(xy)^{1/2} = x^{1/2}y^{1/2}$. For $(\overline{z})^2$ we get the same equations except there is a negative sign for $-2abi$, which for $v \leq 0$ gives us the same answer. □

11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

Proof. Convert z to polar form.

$$\begin{aligned} z &= a + bi \\ &= r\cos(\theta) + r\sin(\theta)i \\ &= r(\cos(\theta) + \sin(\theta)i) \\ &= rw \end{aligned}$$

where $w = \cos(\theta) + \sin(\theta)i$ and $|w| = (\cos(\theta)^2 + \sin(\theta)^2)^{1/2} = 1$. Yes, w and r are always uniquely determined by z because the phase and modulus depend on the complex number. □

12. If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Proof. Using induction with the same method that is used in Theorem 1.33 for proving part (e), which was the base case of n=2.

$$\begin{aligned} |z_1 + z_2 + \cdots + z_n| &\leq |(z_1 + z_2 + \cdots + z_{n-1}) + z_n| \\ &\leq |z_1 + z_2 + \cdots + z_{n-1}| + |z_n| \\ &\leq |z_1| + |z_2| + \cdots + |z_{n-1}| + |z_n|. \end{aligned}$$

□

13. If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

Proof. This is the reverse triangle inequality. Since $x = x - y + y$

$$\begin{aligned} |x| &= |(x - y) + y| \\ &\leq |x - y| + |y| \\ |x| - |y| &\leq |x - y|. \end{aligned}$$

Starting with $y = y - x + x$ we would have arrived at $|y| - |x| \leq |y - x|$. Therefore $||x| - |y|| \leq |x - y|$. □

14. If z is a complex number such that $|z| = 1$, that is such that $z\overline{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

Proof.

$$\begin{aligned} |1 + z|^2 + |1 - z|^2 &= (1 + z)\overline{(1 - z)} + (1 - z)\overline{(1 - z)} \\ &= (1 + z)(1 + \overline{z}) + (1 - z)(1 - \overline{z}) \\ &= 1 + \overline{z} + z + 1 + 1 - \overline{z} - z + 1 \\ &= 4. \end{aligned}$$

[Note: $\overline{z + w} = \overline{z} + \overline{w}$ by Theorem 1.31 (a)] □

15. Under what conditions does equality hold in the Schwarz inequality?

The two sides of the Schwarz inequality are equal when the two vectors are linearly dependent. In other words, when the vectors either point in the same or opposite direction, or if one of them is the zero vector.

16. Suppose $k \geq 3,\ x, y \in \mathbb{R}^k,\ |x - y| = d > 0$, and $r > 0$. Prove:

(a) If $2r > d$, there are infinitely many $z \in \mathbb{R}^k$ such that

$$|z - x| = |z - y| = r.$$

Proof. For $k = 3$, with $z \in \mathbb{R}^3$, a sphere with center $x = (x_0, x_1, x_2)$ and radius r is the locus of all points $z = (z_0, z_1, z_2)$ such that

$$\begin{aligned} r^2 &= (z_0 - x_0)^2 + (z_1 - x_1)^2 + (z_2 - x_2)^2 \\ r^2 &= |z - x|^2 \\ (r^2)^{\frac{1}{2}} &= (|z - x|^2)^{\frac{1}{2}} \\ r &= |z - x| \end{aligned}$$

The same argument shows that $|z - y|$ is the radius of a sphere centered at y. Again, since $d = |x - y|$, we see that d is the radius of a sphere centered at y. Showing this graphically can illuminate the argument. Figure 1 shows a 2D slice of the three spheres in $\mathbb{R}^3$. As we can see, for the case of $2r > d$, the two spheres of equal radius r that are centered at x and y will intesect since $2r > d$ and two of those solutions are z' and z''. We can't see it in Figure 1, but in 3 dimensions, the two intesecting spheres will intersect in a circle and thus there will be an infinite number of solutions z in this case.

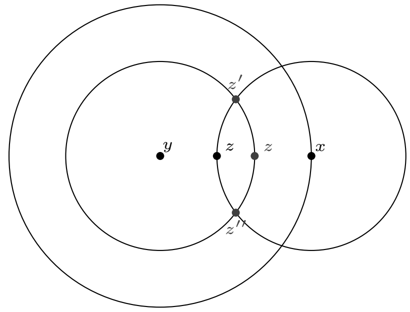

Figure 1: The case $2r > d$ will have two intesecting spheres of radius $|z - x|$ and $|z - y|$.

□

(b) If $2r = d$, there is exactly one such z.

Proof. For this case, we will have the situation where the two spheres of radius $|z - x|$ and $|z - y|$ will have a single solution, namely where the two spheres touch at the single point z as seen in Figure 2.

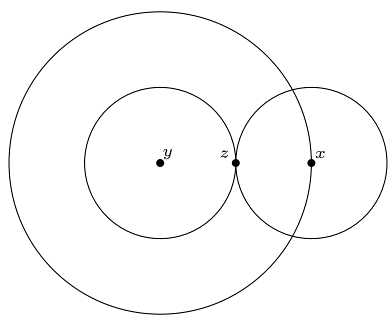

Figure 2: The case $2r = d$ will have two spheres of radius $|z - x|$ and $|z - y|$ touching at a single point z.

□

(c) If $2r < d$, there is no such z.

Proof. For this case, we will have the situation where the two spheres of radius $|z - x|$ and $|z - y|$ will not intersect and therefore there are no solutions.

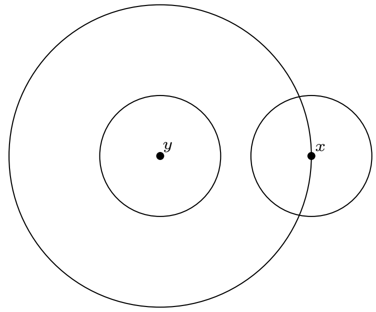

Figure 3: The case $2r < d$ will have two spheres of radius $|z-x|$ and $|z-y|$ that do not intesect.

□

How must these statements be modified if k is 2 or 1?

For $k = 2$ (a) will go from an infinite amount of solutions to just two, for the two points where the circles intesect such as seen in Figure 1. For $k = 1$ (a) will have no solutions. (b) and (c) will still have the same answeres.

17. Prove that

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$$

if $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

Proof. If $x, y \in \mathbb{R}^k$, then by proof of Theorem 1.37 (e),

$$\begin{aligned} |x + y|^2 &= (x + y) \cdot (x + y) \\ &= x \cdot x + 2x \cdot y + y \cdot y \\ |x - y|^2 &= x \cdot x - 2x \cdot y + y \cdot y \end{aligned}$$

Therefore,

$$\begin{aligned} |x + y|^2 + |x - y|^2 &= 2x \cdot x + 2y \cdot y \\ &= 2|x|^2 + 2|y|^2. \end{aligned}$$

This result relates to parallelograms and can be viewed as the sum of the squares of the diagonals are equal to the sum of the squares of the four sides. □

18. If $k \geq 2$ and $x \in \mathbb{R}^k$, prove that there exists $y \in \mathbb{R}^k$ such that $y \neq 0$ but $xy = 0$. Is this also true if $k = 1$?

Proof. If $x = (x_1, x_2)$, let $y = (x_2, -x_1)$. Then $x \cdot y = x_1 x_2 - x_2 x_1 = 0$.

It is not true for $k = 1$ unless we are allowed to have $y = 0$. □

19. Suppose $a \in \mathbb{R}^k, b \in \mathbb{R}^k$. Find $c \in \mathbb{R}^k$ and $r > 0$ such that

$$|x-a| = 2|x-b|$$

if and only if $|x-c| = r$. (*Solution:* $3c = 4b - a, 3r = 2|b-a|$).

Proof. We are given the solution so all we need to do is verify it. We have that

$$\begin{aligned}
|x-a| &= 2|x-b| \\
(|x-a|)^2 &= (2|x-b|)^2 \\
3|x|^2 + 2a\cdot x - 8b\cdot x - |a|^2 + 4|b|^2 &= 0
\end{aligned}$$

$$\begin{aligned}
|x-c| &= r \\
\left|x - \frac{4}{3}b + \frac{1}{3}a\right| &= \frac{2}{3}|b-a| && \text{[from the given solution]} \\
\left(\left|x - \frac{4}{3}b + \frac{1}{3}a\right|\right)^2 &= \left(\frac{2}{3}|b-a|\right)^2 \\
|x|^2 + \frac{2}{3}a\cdot x - \frac{8}{3}b\cdot x - \frac{1}{3}|a|^2 + \frac{4}{3}|b|^2 &= 0 \\
3|x|^2 + 2a\cdot x - 8b\cdot x - |a|^2 + 4|b|^2 &= 0 && \text{[multiply both sides by 3]}
\end{aligned}$$

Therefore, both of these equations are equivalent. □

20. With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.

Proof. Let us construct $\mathbb{R}$ from $\mathbb{Q}$ by using the definition of a *cut* as in the Appendix but without property (III). By removing property (III) from the definition of a cut, the cut α now has a largest member. Let us denote this largest member as u and thus, $\forall p \in \alpha$ we have $p \leq u$. Therefore, u is an upper bound of α and additionally it is the least upper bound since it is the maximum of α. Step 2 of the proof in the Appendix shows us that $\mathbb{R}$ is an ordered set as this step relies on property (II) and not property (III). To show that the ordered set $\mathbb{R}$ has the least-upper-bound property we can follow the same arguments as Step 3 of the proof in the Appendix. However, since we no longer have property (III), this no longer needs to be verified. We still see that $\gamma \in \mathbb{R}$. Furthermore, it is clear that $\alpha \leq \gamma$ for ever $\alpha \in A$ and as was shown in the proof of the Appendix the same arguements hold to show that $\gamma = \sup A$.

Step 4 of the proof in the Appendix similarly will stay the same for $(A1)$, $(A2)$, and $(A3)$, with the part in $(A1)$ needing to verify property (III) ommitted. For $(A4)$ let $O = \{r \mid r \leq 0\}$ and $\alpha \in \mathbb{R}$. By defintion, we can see that O is a cut. I claim $O + \alpha = \alpha$. First, we obviously have $O + \alpha \subseteq \alpha$ since $r + s \leq s$ if $r \leq 0$. Therefore, $r + s \in \alpha$ if $s \in \alpha$. Conversely $\alpha \subseteq O + \alpha$, since each s in α can be written as $0 + s$.

Unfortunately, if $O' = \{r \mid r < 0\}$, there is no element $\alpha \in \mathbb{R}$ such that $\alpha + O' = O$. For $\alpha + O'$ has no largest element. If $x = r + s \in \alpha + O'$, where $r \in \alpha$ and $s \in O'$, there is an element $t \in O'$ with $t > s$, and so $r + t \in \alpha + O'$ and $r + t > s$. Since O has a largest element (namely 0), these two sets cannot be equal. □