Classical Mathematical Logic

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I Classical Propositional Logic

Exercises:

A. Propositions, B. Types

1. Of the following, (i) Which are declarative sentences? (ii) Which contain indexicals? (iii) Which are propositions?

a. Ralph is a dog. $\implies$ (i), (iii)

b. I am 2 meters tall. $\implies$ (i), (ii)

c. Is any politician not corrupt? – None

d. Feed Ralph. $\implies$ None

e. Ralph didn't see George. $\implies$ (i), (iii)

f. Whenever Juney barks, Ralph gets mad. $\implies$ (i), (iii)

g. If anyone should say that cats are nice, then he is confused. $\implies$ (i), (ii)

h. If Ralph should say that cats are nice, then he is confused. $\implies$ (i), (ii)

i. If Ralph should say that cats are nice, then Ralph is confused. $\implies$ (i), (iii)

j. Dogs can contract myxomatosis. $\implies$ (i), (iii)

k. $2 + 2 = 4 \implies$ (i), (iii)

l. $\dfrac{de^x}{dx} = e^x \implies$ (i), (iii)

2. Explain why we cannot take sentence types as propositions if we allow the use of indexicals in our reasoning.

We cannot take sentence types as propositions if we allow the use of indexicals in our reasoning because the truth-values of the sentences will *depend* on a particular situation and what is being referenced.

C. The Connectives of Propositional Logic

1. Classify each of the following as a conjunction, disjunction, negation, conditional, or none Ralph of is those. If a conditional, identify the antecedent and consequent.

a. Ralph is a dog $\wedge$ dogs bark. $\implies$ conjunction

b. Ralph is a dog $\rightarrow$ dogs bark. $\implies$ conditional (antecedent: Ralph is a dog, consequent: dogs bark)

c. $\neg$ (cats bark). $\implies$ negation

d. Cats bark $\vee$ dogs bark. $\implies$ disjunction

e. Cats are mammals and dogs are mammals. $\Longrightarrow$ none

f. Either Ralph is a dog or Ralph isn't a dog. $\Longrightarrow$ none

g. ¬ cats bark → (¬ (cats are dogs)). $\Longrightarrow$ conditional (antecedent: ¬ cats bark, consequent: ¬ (cats are dogs))

h. Cats aren't nice. $\Longrightarrow$ none

i. Dogs bark ∨ (¬ dogs bark). $\Longrightarrow$ conjunction

j. Ralph is a dog. $\Longrightarrow$ none

k. It is possible that Ralph is a dog. $\Longrightarrow$ none

l. Some dogs are not white. $\Longrightarrow$ none

2.

a. Write a sentence that is a negation of a conditional, the antecedent of which is a conjunction. ¬ ((Socrates is a man ∧ men are mortal) → Socrates is mortal).

b. Write a sentence that is a conjunction of disjunctions, each of whose disjuncts is either a negation or has no formal symbols in it.

(An 8 ball is black ∨ (¬ (Elvis is dead))) ∧ ((¬ (the Earth is a planet)) ∨ Dogs are mammals).

3. Write a sentence that might occur in daily speech that is ambiguous but which can be made precise by the use of parentheses, indicating at least two ways to parse it.

I saw someone on the mountain with binoculars while wearing a black shirt.
(I saw)(someone on the mountain with binoculars while wearing a black shirt).
(I saw someone on the mountain with binoculars)(while wearing a black shirt).

4. List at least three words or phrases in English not discussed in the text that are used to form a proposition from one or more propositions and that you believe are important in a study of reasoning.

Some words that some to mind are: therefore, suppose, such that, thus, which shows that, thus we can conclude, proving that, in a similar vein....etc.

These are some of the words that I personally use when constructing premises into arguments.

D. A Formal Language for Propositional Logic

1. Why do we introduce a formal language?

We introduce a rigid formal language to make precise the syntax of the propositions we will study. The reason this is needed is that there is no list of all propositions in English, nor is there a way to generate all of them, as English is not a fixed, formal, static language. A formal language allows us to analyze the *logical forms* of the propositions more easily due to its abstraction.

2. Identify which of the following are formal (unabbreviated) wffs:

a. $(p_1) \vee \neg(p_2)$ $\Longrightarrow$ yes

b. $((p_1) \rightarrow (p_2))$ $\Longrightarrow$ yes

c. $((p_1 \lor p_2) \to (p_2)) \implies$ yes

d. $(\neg(p_1)(p_2) \land (p_1)) \implies$ no

e. $(\neg(\neg(p_1)) \lor \neg(p_1)) \implies$ yes

f. $((\neg(\neg(p_1))) \lor \neg(p_1)) \implies$ yes

g. $((\neg(\neg(p_1))) \lor (\neg(p_1))) \implies$ yes

3. Abbreviate the following wffs according to our conventions on abbreviations:

a. $((((p_1) \to (p_2)) \land (\neg(p_2))) \to (\neg(p_1)))$

$$(p_1 \to p_2) \land \neg p_2 \to \neg p_1$$

b. $((((p_4) \land (p_2)) \lor (\neg(p_6))) \to ((p_7) \to (p_8)))$

$$p_4 \land p_2 \lor \neg p_6 \to p_7 \to p_8$$

4. Write the following informally presented wffs without abbreviations:

a. $(p_1 \lor p_2) \implies ((p_1) \lor (p_2))$

b. $(p_0 \land \neg p_1) \implies ((p_0) \land (\neg(p_1)))$

c. $(p_0 \land \neg\neg p_1) \implies ((p_0) \land (\neg(\neg(p_1))))$

d. $(p_1 \to p_2) \implies ((p_1) \to (p_2))$

e. $(p_1 \to (p_2 \to p_3)) \implies ((p_1) \to ((p_2) \to (p_3)))$

f. $(\neg p_1) \implies (\neg(p_1))$

g. $(\neg\neg p_1) \implies (\neg(\neg(p_1)))$

h. $(\neg\neg\neg\neg p_1) \implies (\neg(\neg(\neg(\neg(p_1)))))$

5. Give an example (using abbreviations) of a formula that is:

a. A conjunction, the conjuncts of which are disjunctions of either atomic propositions or negated atomic propositions.

$$(p_1 \lor \neg p_2) \land (\neg p_3 \lor p_4)$$

b. A conditional whose antecedent is a disjunction of negations and whose consequent is a conditional whose consequent is a conditional.

$$(\neg p_1 \lor \neg p_2) \to (p_3 \to (p_4 \to p_5))$$

6.

a. Calculate the length of the following wffs.

i. $((((p_1) \to (p_2)) \land (\neg(p_2))) \to (\neg(p_1))) \implies 4$

ii. $((((p_4) \land (p_2)) \lor (\neg(p_6))) \to (\neg((p_7) \to (p_8)))) \implies 4$

b. Give an unabbreviated example of a wff of length 5. For a wff of length 5 what is the minimum number of parentheses that can be used? The maximum? For a wff of length 10? (Hint: See Exercise 5 on p. 26 below.)

$$((((p_1) \vee (p_2)) \wedge (p_3)) \rightarrow ((p_5) \wedge (p_6))) \rightarrow (((p_7) \rightarrow (\neg (p_8))))$$

The minimum number of parentheses that can be used is 10? while the maximum is ?. For a wff of length 10 the minimum is ? while the maximum is ?

7. List all subformulas of each of the wffs in Exercise 6.a. What propositional variables appear in them?

We shall split up the subformulas in steps to illustrate the process.

a. $((((p_1) \rightarrow (p_2)) \wedge (\neg (p_2))) \rightarrow (\neg (p_1)))$

- $((p_1 \rightarrow p_2) \wedge (\neg p_2))$, $(\neg p_1)$
- $(p_1 \rightarrow p_2)$, $(\neg p_2)$, $(\neg p_1)$
- (p_1), (p_2), $(\neg p_2)$, $(\neg p_1)$

b. $((((p_4) \wedge (p_2)) \vee (\neg (p_6))) \rightarrow (\neg ((p_7) \rightarrow (p_8))))$

- $((p_4 \wedge p_2) \vee (\neg p_6))$, $(\neg (p_7 \rightarrow p_8))$
- $(p_4 \wedge p_2)$, $(\neg p_6)$, $(\neg (p_7 \rightarrow p_8))$, $(p_7 \rightarrow p_8)$
- $(p_4 \wedge p_2)$, (p_4), (p_2), $(\neg p_6)$, (p_6), $(\neg (p_7 \rightarrow p_8))$, $(p_7 \rightarrow p_8)$, (p_7), (p_8)

Note that in (a.), the negation of p_1, p_2 weren't split up more and this is because we already had these subformulas from the conditional.

8. Distinguish the following: ordinary language, the formal language, a semi-formal language.

Ordinary language is propositional logic without any formalization. The formal language is with formal wffs and the semi-formal language is wffs with some realization applied.

9.

a. Give the realization of the following, using (1) on p. 11:

i. $((p_8 \wedge p_{4318}) \wedge p_7) \rightarrow p_1$

((Cats are nasty ∧ (If Ralph is barking, then Ralph will catch a cat)) ∧ Ralph is barking) → Four cats are sitting in a tree

ii. $(p_0 \wedge p_1) \rightarrow p_2$

(Ralph is a dog ∧ Four cats are sitting in a tree) → Four is a lucky number

iii. $\neg(p_4 \wedge \neg p_5)$

¬(Juney is barking loudly ∧ ¬Juney is barking)

iv. $p_3 \rightarrow \neg\neg p_6$

Dogs bark → ¬¬Dogs bark

v. $\neg(p_{312} \wedge p_7) \wedge \neg p_{317}$

¬(Bill is afraid of dogs ∧ Ralph is barking) ∧ ¬Bill is walking quickly

vi. $p_{312} \wedge p_7 \to \neg p_{317}$

Bill is afraid of dogs ∧ Ralph is barking → ¬Bill is walking quickly

b. The following wff is the realization of what formal wff in realization (1)? (Four cats are sitting in a tree ∧ four is a lucky number) → ¬(If Ralph is barking then he will catch a cat → Howie is a cat) $(p_1 \wedge p_2) \to \neg(p_{4318} \to p_{47})$

c. Exhibit formal wffs of which the following could be taken to be realizations:

i. Cats are nasty → Howie is nasty

$((p_8) \to (p_{10}))$

[where p_8 is from (1) in the textbook and p_{10} ='Howie is nasty']

ii. ¬((Ralph is a dog ∨ ¬¬Ralph barks) ∨ Ralph is a puppet) → no number greater than 4 billion is a perfect square

$\neg(((p_0) \vee (\neg(\neg(p_9)))) \vee (p_{11})) \to (p_{12})$

[where p_0, p_9 are from (1) in the textbook and p_{11} ='Ralph is a puppet' and p_{12} ='no number greater than 4 billion is a perfect square']

E. Classical Propositional Logic

E.1, E.2

1. Why do we take $A \to B$ to be true if A is false?

We take the conditional $A \to B$ to be true if A is false because if A is false then our implication is no longer within context. That is, the *condition* of the conditional is no longer applicable and since the antecedent "does not apply", these cases are treated as vacuously true.

2. What is a truth-functional connective?

A truth-functional connective is a connective that operates semantically as a function of the truth-values of the constituent propositions.

3. For each semi-formal wff below, find an assignment of truth-values to the atomic propositions in realization (1) that will make it have truth-value T, if that is possible.

a. Ralph is barking → cats are nasty

Let p_7='Ralph is barking' and p_8='cats are nasty'.

p_7	p_8	$p_7 \to p_8$
T	T	T
T	F	F
F	T	T
F	F	T

This semi-formal wff will have truth-value T, for the assignments of truth-values to the atomic propositions as seen in rows 1, 3, and 4.

b. (Ralph is a dog ∨ dogs bark) ∧ ¬(Juney is barking)

Let p_0='Ralph is a dog', p_3='dogs bark', and p_5='Juney is barking'.

p_0	p_3	p_5	$(p_0 \vee p_3)$	$\neg(p_5)$	$(p_0 \vee p_3) \wedge \neg(p_5)$
T	T	T	T	F	F
T	F	T	T	F	F
T	T	F	T	T	T
T	F	F	T	T	T
F	T	T	T	F	F
F	F	T	F	F	F
F	T	F	T	T	T
F	F	F	F	T	F

This semi-formal wff will have truth-value T, for the assignments of truth-values to the atomic propositions as seen in rows 3, 4, and 7.

c. Ralph is barking ∧ cats are nasty → Ralph is barking

Let p_7='Ralph is barking', p_8='cats are nasty', and p_{4719}='Ralph is barking'

p_7	p_8	p_{4719}	$p_7 \wedge p_8$	$p_7 \wedge p_8 \rightarrow p_{4719}$
T	T	T	T	T
T	F	T	F	T
T	T	F	T	F
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T
F	T	F	F	T
F	F	F	F	T

This semi-formal wff will have truth-value T, for the assignments of truth-values to the atomic propositions as seen in all rows except row 3. The reason this is the case is that the only time a conditional has truth-value F is when the antecedent is T and the consequent is F.

d. Dogs bark ∨ ∨(dogs bark) [as a realization of $p_6 \vee \neg(p_6)$]

Let p_6='Dogs bark' (and 'dogs bark')

p_6	$\neg p_6$	$p_6 \vee \neg(p_6)$
T	F	T
F	T	T

This semi-formal wff will have truth-value T, for the all assignments of truth-values to the atomic propositions.

e. Dogs bark ∨ ∨(dogs bark) [as a realization of $p_6 \vee \neg(p_3)$]

Let p_6='Dogs bark' and p_3='dogs bark'

p_6	p_3	$\neg p_3$	$p_6 \vee \neg(p_3)$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

This semi-formal wff will have truth-value T, for the assignments of truth-values to the atomic propositions as seen in all rows except row 3.

f. ¬(cats are nasty → ¬¬(cats are nasty))

Let p_8='cats are nasty'

p_8	$\neg\neg p_8$	$p_8 \to \neg\neg p_8$	$\neg(p_8 \to \neg\neg p_8)$
T	T	T	F
F	F	T	F

This semi-formal wff will not have any truth-value T.

g. ((Ralph is a dog ∨ dogs bark)∧ ¬(Ralph is a dog)) → dogs bark

Let p_0='Ralph is a dog' and p_3='dogs bark'

p_0	p_3	$\neg p_0$	$(p_0 \vee p_3)$	$((p_0 \vee p_3) \wedge \neg p_0)$	$((p_0 \vee p_3) \wedge \neg p_0) \to p_3$
T	T	F	T	F	T
T	F	F	T	F	T
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T
F	T	T	T	T	T
F	F	T	F	F	T

This semi-formal wff will have truth-value T, for the all assignments of truth-values to the atomic propositions.

4. Distinguish between a realization and a model.

A realization is an assignment of propositions to some or all of the propositional variables used whereas a model is a realization of the propositional variables as propositions, an assignment of truth-values to those atomic propositions, and the extension of that assignment to all formulas of the semi-formal language via the classical truth-tables.

5. Show by induction on the length of wffs that in a model the valuation plus the truth-tables determine uniquely the truth-value of every compound proposition.

Proof. From a given model, suppose that a wff, say A, has length 1. Then A must be an atomic proposition. Therefore, a valuation of this atomic proposition will be the truth-value that it was assigned in the model (trivial truth-table), which is obviously unique (if it wasn't, it wouldn't be a proposition).

Now, suppose that up to length n, the valuation of A and the truth-tables determine uniquely the truth-value of A. We will now show that this is also the case for a wff of length $n + 1$.

Let the wff A have length $n + 1$. Then, since A is not atomic it must be a compound wff and it must be of the form $(\neg B)$, $(B \to C)$, $(B \wedge C)$, $(B \vee C)$ (among other combinations of B and C) for wffs B, C where the maximum of the numbers assigned to B and C is n [(ii) inductive definition of wff, p. 7]. Since the maximum of the numbers assigned to B and C is n, from the induction hypothesis we know that they both have a unique truth-value determined by the valuation and the truth-tables. Additionally, since the connectives of the model are truth-functionals, we see that the truth-tables for the forms of the wff A mentioned above, will give unique truth-values.

Therefore, in a model the valuation plus the truth-tables determine uniquely the truth-value of every compound proposition. □

E.3

1.

a. What is a tautology in the formal language?

In a formal language, a tautology is a formal wff whose realization in every model evaluates to true.

b. What is a tautology in the semi-formal language?

In a semi-formal language, a tautology is a semi-formal wff that is the realization of some wff that is a tautology.

c. What is a tautology in ordinary English?

A tautology in ordinary English is a proposition that has a straightforward formalization of it into semi-formal English that is a tautology.

2. Exhibit three formal classical tautologies and three formal classical contradictions. Let A be a wff.

tautologies:

$$A \vee \neg A,\ A \to (B \to A),\ A \wedge B \to A$$

contradictions:

$$\neg(A \vee \neg A),\ \neg(A \to (B \to A)),\ \neg(A \wedge B \to A)$$

3.

a. What is a classical semantic consequence?

A classical semantic consequence is a valid inference. That is, a wff and a collection of wffs, say Γ are a semantic consequence if and only if the wff is true whenever all the wffs of Γ are true, for every model.

For example, if we take Γ to be the collection of wffs that have the logical form of the antecedent of modus ponens, then a wff with the logical form of the consequent of modus ponens together with Γ will be a semantic consequence.

b. Exhibit three formal classical semantic consequences.

$$\{p \to q, p\} \vDash q$$
$$\{p \to q, \neg q\} \vDash \neg p$$
$$\{p \vee q, \neg q\} \vDash p$$

F. Formalizing Reasoning

1. What do we mean by 'the logical form of a proposition'?

'the logical form of a proposition' is the wff of the proposition in the formal language.

2. Present and argue for formalizations of the following English connectives:

a. not both A and B

$\neg(A \wedge B)$

Analysis Reading this it is rather easy to see that: 'that it is not the case both A and B'. The formalization follows.

b. when A, B

$A \rightarrow B$

Analysis 'when A' can be seen to be equivalent to 'if A' and therefore 'when A, B' becomes 'if A, B' and from (6) we see that this is equivalent to the conditional formalization.

c. B just in case A

$A \rightarrow B$

Analysis 'just in case A' can be seen to mean the same thing as 'in case A' and from (6), the conditional formalization follows.

d. A, when B

$B \rightarrow A$

Analysis From a. we saw that 'when' can be interpreted as 'if' and therefore we have 'A, if B' and from (6), the formalization follows.

3.

a. Show that a conditional and its contrapositive have the same truth-value.

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

b. Give an example of a conditional and its converse that have different truth-values.

If you won the lottery, you said you would quit your job.

You quit your job.

Therefore, you won the lottery.

Obviously the converse may not *always* be true, when the conditional was. This is an example of *affirming the consequent.*

4. Give two examples not from the text of English connectives that are not truth-functional and show why they are not truth-functional. (Hint: Compare Example 3.)

Two examples of English connectives that are not truth-functional that were not already covered in the textbook are:

'since' and *'as a consequence of'*

These words are not truth-functional due to the same reasons *'because'* is not a truth-functional (see Example 3.). That is, take two propositions that are not causally linked in any regard and use these connectives to infer that the statements are false regardless of the truth-values of the individual propositions. This shows that these connectives are not truth-functional.

5. Formalize and discuss the following in the format above, or explain why they cannot be formalized in classical logic.

a. 7 is not even.

   7 is not even.

   $\neg p$

   Analysis This statement could be viewed as an atomic proposition in and of itself since '7 is not even' can only be true or false. However, since there is a 'not' in the statement, it seems that this would be best viewed as the negation of the atomic proposition '7 is even'. The chosen formalization follows.

b. If 7 is even, then 7 is not odd.

   If 7 is even, then 7 is not odd.

   $p \to \neg\neg p$

   Analysis It is easy to see that '7 is even' is the antecedent for this conditional. The consequent isn't as straight forward. Although we could just take the consequent to be the negation of the statement '7 is odd', if we look closer and note that since numbers can only be even or odd the statement '7 is odd' is the same as the negation of '7 is even'. Formalizing the consequent this way would mean that we actually have a double negation of the antecedent. Since a double negation of a proposition is the proposition itself, this conditional therefore gives us a tautology.

c. Unless Milt has a dog, Anubis is the best-fed dog in Cedar City.

   Unless Milt has a dog, Anubis is the best-fed dog in Cedar City.

   $\neg q \to p$

   Analysis This statement can be re-written as 'Anubis is the best-fed dog in Cedar City unless Milt has a dog' without changing its truth-value. From (6) we see that this has the given formalization where $p =$'Anubis is the best-fed dog in Cedar City.' and $q =$'Milt has a dog'.

d. Horses eat grass because grass is green.

   Not formalizable.

   Analysis Similar to Example 3 in (6) we cannot formalize this statement since 'because' is not a truth-functional connective.

e. It's impossible that $2+2 \neq 4$.

   It is not the case that $2+2 \neq 4$.

   $\neg p$

   Analysis Rewriting 'It's impossible that' with 'It is not the case that' we come to given formalization.

f. Anubis, you know, eats, sleeps, and barks at night.

   Anubis eats and Anubis sleeps and Anubis barks at night.

   $p \wedge q \wedge r$

   Analysis This can be rewritten as 'Anubis eats and Anubis sleeps and Anubis barks at night'. The formalization follows.

g. You can't make an omelette without breaking eggs.

   It is not the case that you can make an omelette without breaking eggs.

   $\neg p$

   Analysis The statement 'make an omelette without breaking eggs' is a proposition as it is either true or false. Rewriting 'You can't' with 'It is not the case that you can', we have the given formalization.

6. Formalize and discuss an example from a mathematics text.

From Rudin's *Principles of Mathematical Analysis, 3rd Ed. p. 37, Theorem 2.33*:

> *Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.*

$p \rightarrow (q \leftrightarrow r)$

Analysis Letting $p =$'$K \subset Y \subset X$', $q =$'K is compact relative to X' and $r =$'K is compact relative to Y' we have the first given formalization.

7. Formalize each of the following inferences. Then evaluate it for validity in classical logic. To do that, attempt to determine if it is possible for the premises to be true and the conclusion false. If it is not possible, then the inference is valid.

a. If Ralph is a cat, then Ralph meows. Ralph is not a cat. So Ralph does not meow.

b. If Bob takes Mary Ann to Jamaica for the holidays, she will marry him. Mary Ann married Bob last Saturday. So Bob must have taken Mary Ann to Jamaica for the holidays.

c. The students are happy if and only if no test is given. If the students are happy, the professor feels good. But if the professor feels good, he won't feel like lecturing, and if he doesn't feel like lecturing a test is given. Therefore, the students are not happy. (*Mates, 1965*)

d. The government is going to spend less on health and welfare. If the government is going to spend less on health and welfare, then either the government is going to cut the Medicare budget or the government is going to slash spending on housing. If the government is going to cut the medicare budget, the elderly will protest. If the government is going to slash spending on housing, then the poor will protest. Therefore, the elderly will protest or the poor will protest.

e. All men are mortal. Socrates is a man. Therefore, Socrates is mortal.

8. Make a list of each term or formal symbol that is introduced or defined in this chapter.

Some smaller items may be missing but this gives a good guideline/list of the main items covered in the chapter:

- Propositions: A written or uttered declarative sentence used in such a way that it is true or false, but not both.
- 'Words/Propositions/Symbols and formulas of the formal language' are types: 'Words/Propositions/Symbols and formulas of the formal language' are types in that any 'words/propositions/symbols/formulas' with the same configuration of letters will be treated as having the same properties of interest to logic.
- Single quotes: Single quotes are used to group words together so that they are considered a single unit. For example, 'The Taj Mahal' has eleven letters.
- Scare quotes: The example in the textbook about getting away with "murder" was a great example (pun intended by the author no doubt).
- Connectives of Propositional Logic:

$\wedge$	and, conjunction [of conjuncts]
$\vee$	or, disjunction [of disjuncts]
$\neg$	it's not the case that, negation
$\rightarrow$	if...then..., conditional [if 'antecedent', then 'consequent']
$\leftrightarrow$	if and only if, biconditional

- Parentheses: Further formal device used for clarity.
- Propositional variables: Variables used in a formal language in place of propositions.
- Formal language: Propositional variables, logical connectives, and parentheses together make the formal language.
- Metavariables: Variables used to stand for any combination of propositional variables and logical connectives. This is a further abstraction to simplify the writing of complex expressions.
- Well-formed formula, or wff: The analogue of a sentence in English within the formal language (see textbook for more details, namely the inductive definition on p. 7).
- Length of a wff: The length of a wff is the number assigned to it in the inductive definition of wffs.
- Schema: Formal wffs with the variables for propositions replaced by metavariables. The skeletons of wffs.
- Subformula: A wff that is a component of another wff within the formal language.
- Realizations: An assignment of propositions to some or all of the propositional variables.
- Realization of a formal wff — semi-formal wff: The formula we get when we replace the propositional variables appearing in the formal wff with the propositions assigned the them.
- Semi-formal language — semi-formal English or formalized English: The collection of realizations of

formal wffs all of whose propositional variables are realized.

- Atomic proposition: wff of length 1. The propositions that are assigned to the propositional variables.
- The classical abstraction: The only properties of a proposition that matter to logic are its form and its truth-value.
- The Fregean assumption: The truth-value of a proposition is determined by its form and the semantic properties of its constituents.
- Truth-functional (connective): A connective is truth-functional if and only if the truth value of any compound proposition when applying this connective is obtained from the truth-values of the constituent propositions.
- The division of form and content: If two propositions have the same semantic properties, then they are indistinguishable in any semantic analysis, regardless of their form.
- Form and meaningfulness: What is grammatical and meaningful is determined solely by form and what primitive parts of speech are taken as meaningful. In particular, given a semi-formal propositional language, ever well-formed formula will be taken to be a proposition.
- Models: A model is a realization of the propositional variables as propositions, an assignment of truth-values to those atomic propositions, and the extension of that assignment to all formulas of the semi-formal language via the classical truth-tables.
- Tautologies: A formal wff whose realization is evaluated as true in every model.
- Contradiction: A proposition or formal wff that is evaluated as false regardless of the truth-values of its atomic constituents.
- Semantic consequence: The notion of semantic consequence formalizes the idea that one proposition follows from another or collection of others, relative to our understanding of the connectives. A valid inference is a semantic consequence (see textbook for more precise definition).
- Semantic equivalence: Two wffs are semantically equivalent if each is a semantic consequence of the other (in every model they have the same truth-value).
- Classical propositional logic: The definition of classical models, of classical validity (tautologies), and of classical semantic consequences together constitute *classical propositional logic.* Also known as classical propositional calculus which is abbreviated **PC**.
- Logical form: The form of a proposition in a formal language, a wff.

Proof by induction

1. the sum of the first n even number is $n \cdot (n+1)$.

Proof. We will assume that we are dealing with the positive even numbers.

An even number is of the form $2 \cdot n$ for integer n. Therefore, the sum of the first n even numbers is of the form

$$2 \cdot 1 + 2 \cdot 2 + \cdots + 2 \cdot n$$

where the nth even number is $2 \cdot n$.

For the base case we have $n = 1$. Thus, $n \cdot (n+1) = 1 \cdot (1+1) = 2$, which is the first even number.

Now, suppose that the sum of the first $n = k$ even numbers is $k \cdot (k+1) = k^2 + k$. Now we will show it is also true for the first $k+1$ even numbers.

For $n = k+1$, the expression becomes

$$\begin{aligned}(k+1) \cdot ((k+1)+1) &= (k+1) \cdot (k+2) \\ &= k^2 + 3k + 2 \\ &= k^2 + k + 2k + 2 \\ &= k^2 + k + 2 \cdot (k+1)\end{aligned}$$

which is the sum of the first k even numbers plus the $(k+1)$th even number.

Therefore, the sum of the first n even number is $n \cdot (n+1)$. □

2. The sum of the first n odd numbers is n^2.

Proof. We will assume that we are dealing with the positive odd numbers.

An odd number is of the form $2 \cdot n + 1$. Therefore, the sum of the first n odd numbers is of the form

$$(2 \cdot 1 + 1) + (2 \cdot 2 + 1) + \cdots + (2 \cdot (n-1) + 1)$$

where the nth (starting from zero) odd number is $2 \cdot (n-1) + 1$.

For the base case we have $n = 1$. Thus, $n^2 = 1^2 = 2 \cdot (1-1) + 1 = 2 \cdot 0 + 1 = 1$, which is the first odd number.

Now, suppose that the sum of the first $n = k$ odd numbers is k^2. Now we will show it is also true for the first $k+1$ odd numbers.

For $n = k+1$, the expression becomes

$$\begin{aligned}(k+1)^2 &= (k+1) \cdot (k+1) \\ &= k^2 + 2k + 2 \\ &= k^2 + 2 \cdot (k+1)\end{aligned}$$

which is the sum of the first k odd numbers plus the $(k+1)$th odd number.

Therefore, the sum of the first n odd numbers is n^2. □

3. $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6} \cdot n \cdot (n+1) \cdot (2 \cdot n + 1)$.

Proof. In this proof we will show that first n sums of squares is equal to $\frac{1}{6} \cdot n \cdot (n+1) \cdot (2 \cdot n + 1)$.

For the base case we have $n = 1$. Thus, $\frac{1}{6} \cdot n \cdot (n+1) \cdot (2n+1) = \frac{1}{6} \cdot 1 \cdot (1+1) \cdot (2 \cdot 1 + 1) = \frac{1}{6} \cdot (2) \cdot (3) = 1$.

Now, suppose that the sum of the first $n = k$ squares is $\frac{1}{6} \cdot k \cdot (k+1) \cdot (2 \cdot k + 1) = \frac{1}{6} \cdot (2k^3 + 3k^2 + k)$. Now we will show it is also true for the first $n = k + 1$ squares.

For $n = k + 1$, the expression becomes

$$\begin{aligned}
\frac{1}{6} \cdot (k+1) \cdot ((k+1)+1) \cdot (2 \cdot (k+1) + 1) &= \frac{1}{6} \cdot (k+1) \cdot (k+2) \cdot (2 \cdot k + 3) \\
&= \frac{1}{6} \cdot (k+1) \cdot (2k^2 + 7k + 6) \\
&= \frac{1}{6} \cdot (2k^3 + 9k^2 + 13k + 6) \\
&= \frac{1}{6} \cdot (2k^3 + 3k^2 + k + [6k^2 + 12k + 6]) \\
&= \frac{1}{6} \cdot (2k^3 + 3k^2 + k) + \frac{1}{6} \cdot (6k^2 + 12k + 6) \\
&= \frac{1}{6} \cdot (2k^3 + 3k^2 + k) + (k^2 + 2k + 1) \\
&= \frac{1}{6} \cdot (2k^3 + 3k^2 + k) + (k+1)^2
\end{aligned}$$

which is the sum of the first k squares plus the $(k+1)$th square.

Therefore, $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6} \cdot n \cdot (n+1) \cdot (2 \cdot n + 1)$. □

4. The minimum number of parentheses that can appear in a wff of length n is $2n$.

Proof. The base case is when we have an atomic wff of length 1. This atomic proposition only has two parentheses surrounding it and this agrees with $2 \cdot n = 2 \cdot 1 = 2$.

Now, suppose that for $n = k$ that the minimum number of parentheses that can appear in a wff of length k is $2k$. Now we will show it is also true for $n = k + 1$.

Let A and B be wffs where the maximum of the numbers assigned to A and B is k. Then, a wff of length $n = k + 1$ will be of the form [(ii) of inductive definition of wff, p. 7]

$$(\neg A), \ (A \to B), \ (A \land B), \ (A \lor B)$$

From the above expressions we can see that $(\neg A)$ only has one wff of maximum number k and therefore a minimum of parenthesis of $2k$ (induction hypothesis) for A itself, plus the 2 surrounding parentheses to give us $2k + 2 = 2 \cdot (k + 1)$ parentheses.

Therefore, the minimum number of parentheses that can appear in a wff of length n is $2n$. □

5. The maximum number of parentheses that can appear in a wff of length n is $2^{n+1} - 2$.

Proof. The base case is when we have an atomic wff of length 1. This atomic proposition only has two parentheses surrounding it and this agrees with $2^{1+1} - 2 = 2^2 - 2 = 4 - 2 = 2$.

Now, suppose that for $n = k$ that the maximum number of parentheses that can appear in a wff of length k is $2^{k+1} - 2$. Now we will show it is also true for $n = k + 1$.

Let A and B be wffs where the maximum of the numbers assigned to A and B is k. Then, a wff of length $n = k + 1$ will be of the form [(ii) of inductive definition of wff, p. 7]

$$(\neg A),\ (A \to B),\ (A \wedge B),\ (A \vee B)$$

From the above expressions, we can see that any expression that contains both A and B will have two wffs each of maximum number k. Therefore, we will have $(2^{k+1} - 2) + (2^{k+1} - 2)$ parentheses for A and B and including the outer two parentheses this will become $2 \cdot (2^{k+1} - 2) + 2 = 2^{k+2} - 4 + 2 = 2^{(k+1)+1} - 2$ parentheses.

Therefore, the maximum number of parentheses that can appear in a wff of length n is $2^{n+1} - 2$. $\square$