Topology of Numbers
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Chapter 0 - Preview

Exercises:

1.

(a) Make a list of the 16 primitive Pythagorean triples (a, b, c) with $c \leq 100$, regarding (a, b, c) and (b, a, c) as the same triple.

The textbook proves that all primitive Pythagorean triples are obtained from the formula $(a, b, c) = (2pq, p^2 - q^2, p^2 + q^2)$ by letting p and q range over all positive integers with $p > q$, such that p and q have no common factor and are of opposite parity (i.e., one of them is *even* and the other is *odd*). We will use the formula from this proposition in constructing a list of the 16 primitive Pythagorean triples (a, b, c) with $c \leq 100$, regarding (a, b, c) and (b, a, c) as the same triple.

(p, q)	(a, b, c)
$(2, 1)$	$(4, 3, 5)$
$(3, 2)$	$(12, 5, 13)$
$(4, 1)$	$(8, 15, 17)$
$(4, 3)$	$(24, 7, 25)$
$(5, 2)$	$(20, 21, 29)$
$(5, 4)$	$(40, 9, 41)$
$(6, 1)$	$(12, 35, 37)$
$(6, 5)$	$(60, 11, 61)$
$(7, 2)$	$(28, 45, 53)$
$(7, 4)$	$(56, 33, 65)$
$(7, 6)$	$(84, 13, 85)$
$(8, 1)$	$(16, 63, 65)$
$(8, 3)$	$(48, 55, 73)$
$(8, 5)$	$(80, 39, 89)$
$(9, 2)$	$(36, 77, 85)$
$(9, 4)$	$(72, 65, 97)$

(b) How many more would there be if we allowed nonprimitive triples?

Instead of computing this by hand, let's write use Python and our computers to compute this for us. (Note — here we are using the programming language `https://www.python.org/` and the interactive shell `https://ipython.org/` but this code can be adapted to a programming language of your choice as it is relatively straight forward):

```
In [1]: def a(p,q):
   ...:     return 2*p*q
   ...:

In [2]: def b(p,q):
   ...:     return p**2-q**2
   ...:

In [3]: def c(p,q):
   ...:     return p**2+q**2
   ...:

In [4]: triples = []
```

```
In [5]: for p in range(2,10):
   ...:     for q in range(1,p):
   ...:         if c(p,q) <= 100:
   ...:             t = [a(p,q), b(p,q), c(p,q)]
   ...:             ft = [b(p,q), a(p,q), c(p,q)]
   ...:             if t not in triples and ft not in triples:
   ...:                 triples.append(t)
   ...:         for n in range(2,21):
   ...:             if n*c(p,q) <= 100:
   ...:                 t = [n*a(p,q), n*b(p,q), n*c(p,q)]
   ...:                 ft = [n*b(p,q), n*a(p,q), n*c(p,q)]
   ...:                 if t not in triples and ft not in triples:
   ...:                     triples.append(t)
   ...:
   ...:

In [6]: triples
Out[6]:
[[4, 3, 5],
 [8, 6, 10],
 [12, 9, 15],
 [16, 12, 20],
 [20, 15, 25],
 [24, 18, 30],
 [28, 21, 35],
 [32, 24, 40],
 [36, 27, 45],
 [40, 30, 50],
 [44, 33, 55],
 [48, 36, 60],
 [52, 39, 65],
 [56, 42, 70],
 [60, 45, 75],
 [64, 48, 80],
 [68, 51, 85],
 [72, 54, 90],
 [76, 57, 95],
 [80, 60, 100],
 [12, 5, 13],
 [24, 10, 26],
 [36, 15, 39],
 [48, 20, 52],
 [60, 25, 65],
 [72, 30, 78],
 [84, 35, 91],
 [8, 15, 17],
 [16, 30, 34],
 [24, 45, 51],
 [32, 60, 68],
 [40, 75, 85],
 [24, 7, 25],
 [48, 14, 50],
 [72, 21, 75],
 [96, 28, 100],
 [20, 21, 29],
 [40, 42, 58],
 [60, 63, 87],
 [40, 9, 41],
 [80, 18, 82],
 [12, 35, 37],
 [24, 70, 74],
 [60, 11, 61],
 [28, 45, 53],
 [56, 33, 65],
 [84, 13, 85],
 [16, 63, 65],
 [48, 55, 73],
```

```
 [80, 39, 89],
 [36, 77, 85],
 [72, 65, 97]]

In [7]: len(triples)
Out[7]: 52
```

For $c \leq 100$ we have 52 Pythagorean triples.

(c) How many triples (primitive or not) are there with $c = 65$?

```
In [8]: len([t for t in triples if t[2] == 65])
Out[8]: 4
```

From the output of part (b) we see that there are 4 triples with $c = 65$.

2.

(a) Find all the positive integer solutions of $x^2 - y^2 = 512$ by factoring $x^2 - y^2$ as $(x+y)(x-y)$ and considering the possible factorizations of 512.

Since $512 = 2^9$ we must have that $(x+y)$ and $(x-y)$ are powers of two whose product is 512. That is:

$$x^2 - y^2 = 512$$
$$x^2 - y^2 = (x+y)(x-y) = 512$$
$$x^2 - y^2 = (x+y)(x-y) = 512 = 2^9$$

Let $m + n = 9$ so that $(x+y)(x-y) = 2^{m+n} = 2^m 2^n$ and hence $(x+y) = 2^m$ and $(x-y) = 2^n$. Since $(x+y) > (x-y)$ we have that $2^m > 2^n$ and therefore $m > n$. There are only four ways to add up two positive integers equal to 9, and these numbers, m and n, will lead us to the solutions.

$$\begin{matrix} 8+1 \\ 7+2 \\ 6+5 \\ 5+4 \end{matrix} \quad \Longrightarrow \quad \begin{matrix} 2^{8+1} \\ 2^{7+2} \\ 2^{6+3} \\ 2^{5+4} \end{matrix} \quad \Longrightarrow \quad \begin{matrix} 2^8 2^1 \\ 2^7 2^2 \\ 2^6 2^3 \\ 2^5 2^4 \end{matrix} \quad \Longrightarrow \quad \begin{matrix} 256 \cdot 2 \\ 128 \cdot 4 \\ 64 \cdot 8 \\ 32 \cdot 16 \end{matrix}$$

Therefore, all the positive integer solutions, (x, y), of $x^2 - y^2 = 512$ are:

(129, 127), (66, 62), (36, 28), and (24, 8)

(b) Show that the equation $x^2 - y^2 = n$ has only a finite number of integer solutions for each value of $n > 0$.

We need to show that there are not an infinite amount of solutions.

Factoring $x^2 - y^2 = n$ as $(x+y)(x-y) = n$, with n a finite number, shows us that there can only be a finite number of combinations (x, y) that are integer solutions to this equation.

Therefore, the equation $x^2 - y^2 = n$ has only a finite number of integer solutions for each value of $n > 0$ as there are not an infinite amount of solutions.

(c) Find a value of $n > 0$ for which the equation $x^2 - y^2 = n$ has at least 100 different positive integer solutions.

From the solution to part (a) we saw that for a number n that is a power of two, say $n = 2^t$ to be general, with an odd exponent we would have $\frac{t-1}{2}$ solutions to the equation $x^2 - y^2 = n = 2^t$. For an even exponent we would have $\frac{t}{2}$ solutions (use the same argument demonstrated in part (a) to arrive at this conclusion). Therefore, $n = 2^{200}$, as it has an even exponent, would have at least 100 different positive integer solutions.

3.

(a) Show that there are only a finite number of Pythagorean triples (a, b, c) with a equal to a given number n.

From the formulas for Pythagorean triples we can see that as we iterate over p and q to generate the triples that there can only be a finite number of Pythagorean triples (a, b, c) with $a = 2pq$ equal to a given number n due to the fact that p and q are growing in magnitude as we generate the triples. Thus, a given n for a will get passed at some point as more and more triples are generated.

(b) Show that there are only a finite number of Pythagorean triples (a, b, c) with c equal to a given number n.

Same argument as above but this time applied to $c = p^2 + q^2$.

4. Find an infinite sequence of primitive Pythagorean triples where two of the numbers in each triple differ by 2.

The sequence created from $n + \frac{4n+3}{4n+4}$ such that the improper fraction's numerator and denominator are the sides a and b and then the hypotenuse will always be a difference of 2 with the side a. Some examples:

$$\begin{aligned}
1 + \frac{7}{8} &= \frac{15}{8} \implies (15, 8, 17) && [n = 1] \\
2 + \frac{11}{8} &= \frac{35}{12} \implies (35, 12, 37) && [n = 2] \\
3 + \frac{15}{16} &= \frac{63}{16} \implies (63, 16, 65) && [n = 3] \\
&\vdots
\end{aligned}$$

5. Find a right triangle whose sides have integer lengths and whose acute angles are close to 30 and 60 degrees by first finding the irrational value of r that corresponds to a right triangle with acute angles exactly 30 and 60 degrees, then choosing a rational number close to this irrational value of r.

A triangle with acute angles exactly 30 and 60 degrees has its sides in the proportion $1 : \sqrt{3} : 2$. Since $\sqrt{3} \approx 1.732$ we can see that the ratio of the two shorter legs is approximately this amount (due to the unity of the other leg). Taking a look at some Pythagorean triples we can see that $(95, 168, 193)$ has the ratio of $\frac{168}{95} \approx 1.768$, for the two shorter legs. This right triangle has, $\theta = \arccos \frac{95}{193} \approx 60.5$ degrees. Thus, we have a right triangle that has angles approximately 60.5 and 29.5 degrees for its acute angles.

6. Find a right triangle whose sides have integer lengths and where one of the two shorter sides is approximately twice as long as the other, using a method like the one in the preceding problem. (One possible answer might be the $(8, 15, 17)$ triangle, or a triangle similar to this, but you should do better than this.)

In a similar vein to the last problem, if we look at a right triangle that has sides of proportion $1:2:\sqrt{5}$ we see that $\sqrt{5}\approx 2.23$. Once again, taking the ratio $\frac{2.23}{1}=2.23$ and looking at Pythagorean triples for a similar ratio among the hypotenuse and one of the shorter legs we see the triple $(105,208,233)$ has the ratio $\frac{233}{105}\approx 2.22$ and additionally, it is also easy to see that the shortest leg is almost twice as long as the other.

7. Find a rational point on the sphere $x^2+y^2+z^2=1$ whose x, y, and z coordinates are nearly equal.

The formulas for rational points on the unit sphere are:

$$x=\frac{2u}{u^2+v^2+1} \quad y=\frac{2v}{u^2+v^2+1} \quad z=\frac{u^2+v^2-1}{u^2+v^2+1}$$

The question doesn't specify what level of precision is asked for but we can see that if we set $u=v=\frac{11}{8}$ that the numerators become $\frac{11}{4}=2.75$ for x and y, while for z it is $\frac{89}{32}\approx 2.781$. Thus, we have the rational point $\left(\frac{88}{153},\frac{88}{153},\frac{89}{153}\right)$.

8.

(a) Derive formulas that give all the rational points on the circle $x^2+y^2=2$ in terms of a rational parameter m, the slope of the line through the point $(1,1)$ on the circle. (The value $m=\infty$ should be allowed as well, yielding the point $(1,-1)$.) The calculations may be a little messy, but they work out fairly nicely in the end to give

$$x=\frac{m^2-2m-1}{m^2+1}, \quad y=\frac{-m^2-2m+1}{m^2+1}$$

Starting with the equation $y-1=m(x-1)$ and solving for y we find $y=mx-m+1$. Now we plug this into $x^2+y^2=2$ so that:

$$\begin{aligned} x^2+(mx-m+1)^2&=2\\ x^2+m^2x^2-2m^2x+2mx+m^2-2m+1&=2\\ (m^2+1)x^2+(2m-2m^2)x+(m^2-2m-1)&=0 \end{aligned}$$

Then, using the quadratic formula to solve for x we have:

$$x=\frac{2m^2-2m\pm\sqrt{4(m-m^2)^2-4(m^2+1)(m^2-2m-1)}}{2(m^2+1)}$$
$$x=\frac{2m^2-2m\pm\sqrt{4m^2-8m^3+4m^4-4m^4+8m^3+8m+4}}{2(m^2+1)}$$
$$x=\frac{2m^2-2m\pm\sqrt{4m^2+8m+4}}{2(m^2+1)}$$
$$x=\frac{2m^2-2m\pm 2\sqrt{m^2+2m+1}}{2(m^2+1)}$$
$$x=\frac{2m^2-2m\pm 2\sqrt{(m+1)^2}}{2(m^2+1)}$$
$$x=\frac{2m^2-2m\pm 2(m+1)}{2(m^2+1)}$$

$$
\begin{aligned}
x &= \frac{m^2 - m \pm (m+1)}{m^2+1} \\
x &= \frac{m^2 - 2m + 1}{m^2 - 1} && \text{[for } -\text{]} \\
x &= 1 && \text{[for } +\text{]}
\end{aligned}
$$

Since we started from the rational point $(1, 1)$ we will use the solution with the negative square root of the quadratic formula. Plugging this back into the equation for y we have that:

$$
\begin{aligned}
y &= mx - m - 1 \\
y &= m\left(\frac{m^2 - 2m - 1}{m^2+1}\right) - m - 1 \\
y &= \left(\frac{m^3 - 2m^2 - m}{m^2+1}\right) - m - 1 \\
y &= \left(\frac{m^3 - 2m^2 - m - m(m^2+1) + m^2 + 1}{m^2+1}\right) \\
y &= \left(\frac{m^3 - 2m^2 - m - m^3 - m + m^2 + 1}{m^2+1}\right) \\
y &= \frac{-m^2 - 2m + 1}{m^2+1}
\end{aligned}
$$

which gives us the two derived formulas.

(b) Using these formulas, find five different rational points on the circle in the first quadrant, and hence five solutions of $a^2 + b^2 = 2c^2$ with positive integers a, b, c.

Five such rational points are:

$$
\left(\frac{137}{97}, \frac{7}{97}\right), \left(\frac{7}{5}, \frac{1}{5}\right), \left(\frac{89}{65}, \frac{23}{65}\right), \left(\frac{17}{13}, \frac{7}{13}\right), \left(\frac{49}{41}, \frac{31}{41}\right)
$$

(c) The equation $a^2 + b^2 = 2c^2$ can be rewritten as $c^2 = 1/2\left(a^2 + b^2\right)$, which says that c^2 is the average of a^2 and b^2, or in other words, the squares a^2, c^2, b^2 form an arithmetic progression. One can assume $a < b$ by switching a and b if necessary. Find four such arithmetic progressions of three increasing squares where in each case the three numbers have no common divisors.

Four such progressions for coprime tuples (a^2, c^2, b^2) are:

$$
\begin{aligned}
(1, 5, 9) &\to 1^2 + 3^2 = 1 + 9 \implies \frac{1}{2} \cdot 10 = 5 \\
(9, 17, 25) &\to 3^2 + 5^2 = 9 + 25 \implies \frac{1}{2} \cdot 34 = 17 \\
(9, 29, 49) &\to 3^2 + 7^2 = 9 + 49 \implies \frac{1}{2} \cdot 58 = 29 \\
(25, 37, 49) &\to 5^2 + 7^2 = 25 + 49 \implies \frac{1}{2} \cdot 74 = 37
\end{aligned}
$$

9.

(a) Find formulas that give all the rational points on the upper branch of the hyperbola $y^2 - x^2 = 1$.

A single rational point for the upper branch is $(0, 1)$ as $1^2 - 0^2 = 1$. From this rational point we can see that we have a slope-intercept form of the linear equation through this point of $y = mx + 1$. If we denote the point where the line intersects the x-axis by $(r, 0)$, then $m = -\frac{1}{r}$ so the equation for the line can be rewritten as $y = 1 - \frac{x}{r}$ (this is similar to the derivation in the textbook for the unit circle, cf. p. 2). Plugging this equation for y back into the equation of the hyperbola we see that:

$$\begin{aligned}
y^2 - x^2 &= 1 \\
\left(1 - \frac{x}{r}\right)^2 - x^2 &= 1 \\
\left(1 - \frac{x}{r}\right)\left(1 - \frac{x}{r}\right) - x^2 &= 1 \\
1 - \frac{2x}{r} + \frac{x^2}{r^2} - x^2 &= 1 \\
\left(\frac{1}{r^2} - 1\right)x^2 - \frac{2x}{r} &= 0 \\
\left(\frac{1}{r^2} - 1\right)x^2 &= \frac{2x}{r} \\
\left(\frac{1}{r^2} - 1\right)x &= \frac{2}{r} \\
\left(\frac{1 - r^2}{r^2}\right)x &= \frac{2}{r} \\
(1 - r^2)x &= 2r \\
x &= \frac{2r}{1 - r^2}
\end{aligned}$$

Taking this equation for x we can now plug it back into the slope-intercept form of the linear equation $y = 1 - \frac{x}{r}$ so that we get:

$$\begin{aligned}
y &= 1 - \frac{x}{r} \\
y &= 1 - \frac{\frac{2r}{1 - r^2}}{r} \\
y &= 1 - \frac{2}{1 - r^2} \\
y &= \frac{1 - r^2 - 2}{1 - r^2} \\
y &= \frac{-1 - r^2}{1 - r^2}
\end{aligned}$$

(b) Can you find any relationship between these rational points and Pythagorean triples?

Taking the above formulas for x and y we can plug in rational p and q in place of r similar to the textbook's derivation of the Pythagorean triples from the parameterization of the unit circle. You will end up with equations:

$$x = \frac{2pq}{q^2 - p^2} \qquad y = \frac{-q^2 - p^2}{q^2 - p^2}$$

such that a triple is:

$$(2pq, -q^2 - p^2, q^2 - p^2) = (a, -c, -b)$$

Thus, we have rational points of the hyperbola giving solutions to permuted Pythagorean triples!

10.

(a) Show that the equation $x^2 - 2y^2 = \pm 3$ has no integer solutions by considering this equation modulo 8.

As mentioned in the textbook, the squares mod 8 are $0^2 = 0$, $(\pm 1)^2 = 1$, $(\pm 2)^2 = 4$, $(\pm 3)^2 = 9 \equiv 1$, and $4^2 = 16 \equiv 0$, so the squares of even numbers are 0 and 4 mod 8 and the squares of odd numbers are 1 mod 8.

Therefore, the numbers we have overall are 0, 1, and 4. We need the right hand side to equal ± 3 but any combination of these numbers cannot be realized in $x^2 - 2y^2 = \pm 3$.

$$\begin{aligned} 0 - 2(0) &= 0 \\ 0 - 2(1) &= -2 \\ 0 - 2(4) &= -8 \\ 1 - 2(0) &= 1 \\ 1 - 2(1) &= -1 \\ 1 - 2(4) &= -7 \\ 4 - 2(0) &= 4 \\ 4 - 2(1) &= 2 \\ 4 - 2(4) &= -4 \end{aligned}$$

Therefore, the equation $x^2 - 2y^2 = \pm 3$ has no integer solutions by considering this equation modulo 8.

(b) Show that there are no primitive Pythagorean triples (a, b, c) with a and b differing by 3.

There are probably other ways to show this but we will note the argument in the textbook (cf. p. 6) that shows a and b cannot be of the form $4k + 2$ and rather, they are of the form $2k + 1$ and $4k$.

Let $k = 1$, then $a = 4k = 4(1) = 4$ and $b = 2k + 1 = 2(1) + 1 = 3$. If we were able to have a or b of the form $4k + 2$ then we could have $a = 4k + 2 = 4(1) + 2 = 6$ which shows that $a - b = 3$. However, this shows that a and b differ by 3 which is not allowed as the form $4k + 2$ is not allowed.

Therefore, there are no primitive Pythagorean triples (a, b, c) with a and b differing by 3.

This can also be answered by noticing that since $a = 2pq$ and $b = p^2 - q^2$, we can write this as:

$$\begin{aligned} b - a &= p^2 - q^2 - 2pq \\ &= p^2 - 2pq - q^2 \\ &= (p - q)^2 - 2q^2 \end{aligned}$$

and with a change of variables as is done in the textbook for this quadratic form, we see that we have $x^2 - 2y^2$, which is the same as in part (a) above and was shown not to have any solutions.

11. Show there are no rational points on the circle $x^2 + y^2 = 3$ using congruences modulo 3 instead of modulo 4.

First, note that the squares mod 3 are $0^2 = 0$, $(\pm 1)^2 = 1$, $(\pm 2)^2 = 4 \equiv 1$, which shows that even squares are either 0 or 1 while odd squares are 1.

If the circle $x^2 + y^2 = 3$ had a rational point, this would yield a solution of the equation $a^2 + b^2 = 3c^2$ and in the same vein as the argument in the textbook, we see that a, b, and c have no common factor so a and b cannot both be even. The left side of the equation is either $0 + 1$, $1 + 0$, or $1 + 1$ (since a and b are not both even). Thus, the left side of the equation is either 1 or 2 mod 3. However, the right side is either $3 \cdot 0$ or $3 \cdot 1$ mod 4.

Therefore, there are no rational points on the circle $x^2 + y^2 = 3$ using congruences modulo 3 instead of modulo 4.

12. Show that for every Pythagorean triple (a, b, c) the product abc must be divisible by 60. (It suffices to show that abc is divisible by 3, 4, and 5.)

The formula for producing the Pythagorean triples once again is:

$$(a, b, c) = (2pq, p^2 - q^2, p^2 + q^2)$$

where $p > q$, they are both coprime and they have opposite parity. Then, the product abc is:

$$2pq(p^2 - q^2)(p^2 + q^2)$$

which can also be written as

$$2pq(p^4 - q^4)$$

Now we will show that 3, 4, and 5 divide these products using congruences.

For 3:

The squares mod 3 are 0 and 1. If $3 \mid pq$, we are done. If not, then $3 \nmid p$ and $3 \nmid q$. Since $3 \mid p^2 - 1$ and $3 \mid q^2 - 1$ we see that $3 \mid (p^2 - 1) - (q^2 - 1) = (p^2 - q^2)$.

For 4:

Since $p + q$ is odd (due to them having opposite parity), then one of p, q is even and therefore $4 \mid 2pq$.

For 5:

The squares mod 5 are 0, 1 and 4. If $5 \mid pq$, we are done. If not, then $5 \nmid p$ and $5 \nmid q$. Since $5 \mid p^4 - 1$ and $5 \mid q^4 - 1$ we see that $5 \mid (p^4 - 1) - (q^4 - 1) = (p^4 - q^4)$.

The above cases show that 3, 4, and 5 divide the factors of abc.

Therefore, for every Pythagorean triple (a, b, c) the product abc must be divisible by 60.

13. Using congruences modulo 8 show that primitive solutions of $a^2 + b^2 + c^2 = d^2$ must have d odd and must have two of a, b, c even and the other odd.

We can assume that a, b, c, and d do not have a common factor. Then a, b, and c cannot all be even as that would mean that d would also be even and then they would have the common factor of 2. Therefore, d must be odd.

Since:

$$\begin{gathered}\text{even number} + \text{even number} = \text{even number}\\ \text{odd number} + \text{odd number} = \text{even number}\\ \text{odd number} + \text{even number} = \text{odd number}\end{gathered}$$

If d is odd that means that a, b, and c are either all odd or that two of them are even and one is odd. Once again, the squares mod 8 are $0^2 = 0$, $(\pm 1)^2 = 1$, $(\pm 2)^2 = 4$, $(\pm 3)^2 = 9 \equiv 1$, and $4^2 = 16 \equiv 0$, so the squares of even numbers are 0 and 4 mod 8 and the squares of odd numbers are 1 mod 8. Therefore, the left hand side could be either $1 + 1 + 1$ (all odd), $4 + 4 + 1$ (two even, one odd), $0 + 4 + 1$ (two even, one odd), or $0 + 0 + 1$ (two even, one odd). Since the right hand side is odd we know that we have $1 \cdot 1$ and we see that the left hand side cannot be all odd and therefore must be one odd and the other two even. Among these choices we also see that we must either have $4 + 4 + 1$ or $0 + 0 + 1$.

Therefore, using congruences modulo 8 we have shown that primitive solutions of $a^2 + b^2 + c^2 = d^2$ must have d odd and must have two of a, b, c even and the other odd.